Logique

TD n°5

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Exercise 1 : Gödel's construction

In the lecture we have seen a non-constructive argument showing that there is a closed formula G such that neither G nor $\neg G$ are provable. The goal of this exercise is to build such a formula.

Let f be the recursive function such that $f(n,p,q) = 1$ if $n = \ulcorner \pi \urcorner$, $p = \ulcorner A \urcorner$, and π is a proof in PA of $A\{w \to \underline{q}\}$, else $f(n,p,q) = 0$.

Let $F[x_1, x_2, x_3, y]$ be the proposition representing f and $T \overset{\mathsf{def}}{=} \forall x.\ \neg F[x, w, w, \underline{1}]$ with a free variable w. Let $m = \ulcorner T \urcorner$ and G the closed proposition $T\{w \to \underline{m}\}$.

1. Show that if G is provable in PA, then
   (a) $\mathcal{M}_\mathbb{N} \models G$
   (b) for all $n \in \mathbb{N}$, $\mathcal{M}_\mathbb{N} \not\models F[\underline{n}, \underline{m}, \underline{m}, \underline{1}]$
   (c) for all n, $f(n,m,m) = 0$
   (d) the proposition $T\{w \to \underline{m}\}$ is not provable in PA
   (e) the proposition G is not provable in PA
2. Show that if $\neg G$ is provable in PA, then so does G.
3. Conclude that PA is incomplete.

Exercise 2 : Pavages

On rappelle le problème de pavage de Wang (1961):

- Donnée: un ensemble fini de tuiles $T = \{T_i \mid 1 \leq i \leq n\}$ et des relations de compatibilité horizontales et verticales $H, V \subseteq T^2$.
- Question: est-il possible de paver le plan avec ces tuiles (sans rotations des tuiles) en respectant les compatibilités, i.e. existe-t-il $t : \mathbb{N}^2 \to T$ tel que pour tout i, j, $t(i,j)\ H\ t(i+1,j)$ et $t(i,j)\ V\ \ t(i,j+1)$?

Ce problème a été montré indécidable par Berger en 1964.

1. Montrer que si une théorie est incohérente alors elle admet un sous-ensemble fini incohérent. (C'est la propriété de compacité, qui est proche du problème de la complétude des systèmes de preuve.)

2. En déduire qu'une instance du problème de pavage de Wang permet de paver le quart de plan ssi elle permet de paver tout carré fini. (Considérer un encodage du problème de pavage de Wang en logique propositionnelle, utilisant un nombre infini de formules et de symboles de prédicat.)
3. En encodant de nouveau le problème de pavage, montrer que la validité est indécidable dans le calcul des prédicats sans symboles de fonction, et avec des prédicats d'arité au plus deux.

Exercise 3 : A sequence of theories

On a vu qu'il existe une proposition qui n'est ni prouvable ni réfutable dans l'arithmétique de Peano. On peut donc ajouter cette proposition, ou son contraire, pour obtenir une théorie étendue admettant les entiers naturels comme modèle. En répétant l'argument une obtient une séquence de théories comme dans cet exercice; une situation qu'on retrouve aussi dans des arguments de complétude.

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ a sequence of theories such that for every n, $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ and there is a structure $\mathcal{M}_n$ such that $\mathcal{M}_n \vDash \mathcal{T}_n$ but $\mathcal{M}_n \nvDash \mathcal{T}_{n+1}$. Let $\mathcal{T} = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$.

1. Show that $\mathcal{T}$ is consistent.
2. Prove that there is no finite subtheory $\mathcal{T}'$ of $\mathcal{T}$ such that for all structure $\mathcal{M}$, $\mathcal{M} \models \mathcal{T}$ iff $\mathcal{M} \models \mathcal{T}'$.

Exercise 4 : Monadic predicate calculus

Let $\mathcal{F} = \emptyset$ and $\mathcal{P} = \{P_1(1), \ldots, P_n(1)\}$.

1. Prove that for all structure $\mathcal{M}$, there is a structure $\mathcal{N}$ of domain of size smaller than 2^n and a surjective function $f : \mathcal{D}_{\mathcal{M}} \to \mathcal{D}_{\mathcal{N}}$ such that for every formula ϕ of free variables $x_1, \ldots, x_m$ and parameters $a_1, \ldots, a_m \in \mathcal{D}_{\mathcal{M}}^m$:
$$\mathcal{M}, \{x_1 \to a_1, \ldots, x_m \to a_m\} \models \phi \text{ iff } \mathcal{N}, \{x_1 \to f(a_1), \ldots, x_m \to f(a_m)\} \models \phi$$
2. Conclude that if ϕ is a closed formula, $\vdash \phi$ is provable iff ϕ is satisfied in every structure of domain smaller than 2^n. What is the consequence of this result?
3. Extend this argument when unary function symbols are added to the language.

Les exercices ci-dessous ne sont pas à voir obligatoirement.

Exercise 5 : A strange formula

In this exercise you can use the theorem of Mayasevitch, that is that the set of closed provable propositions in arithmetic of the form $\exists x_1 \ldots \exists x_m (t = u)$ is undecidable.

1. Show that there exists a closed proposition A of the form $\exists x_1 \ldots \exists x_m (t = u)$ such that neither A nor $\neg A$ are provable in arithmetic.

2. Show that the proposition $\forall x_1 \ldots \forall x_m \neg(t = u)$ is not provable.
3. Show that if a is a closed term of arithmetic, then there exists $n \in \mathbb{N}$ such that $a = \underline{n}$ is provable.
4. Show that if n and p are two natural numbers, then either $\underline{n} = \underline{p}$ or $\neg(\underline{n} = \underline{p})$ is provable.
5. Show that if a and b are two closed terms of arithmetic, then the proposition $a = b$ is provable or $\neg(a = b)$ is provable.
6. Let $t = u$ be an equation of variables among $x_1, \ldots, x_m$, and $p_1, \ldots, p_m$ natural numbers such that the proposition $(t = u)\{x_1 \to \underline{p_1}, \ldots, x_m \to \underline{p_m}\}$ is provable. Show that the proposition $\exists x_1 \ldots \exists x_m (t = u)$ is provable.

   Show that if $\exists x_1 \ldots \exists x_m (t = u)$ is not provable then for all natural numbers $p_1, \ldots, p_m$, $(t = u)\{x_1 \to \underline{p_1}, \ldots, x_m \to \underline{p_m}\}$ is not provable.
7. Show that if the proposition $\exists x_1 \ldots \exists x_m (t = u)$ is not provable, then for all natural numbers $p_1, \ldots, p_m$, then $\neg(t = u)\{x_1 \to \underline{p_1}, \ldots, x_m \to \underline{p_m}\}$ is provable.
8. Show that there is a proposition ϕ of the form $\neg(t = u)$ of variables among $x_1, \ldots, x_m$ and such that:
   - for all natural numbers $p_1, \ldots, p_m$, the proposition $\phi\{x_1 \to \underline{p_1}, \ldots, x_m \to \underline{p_m}\}$ is provable.
   - the proposition $\forall x_1 \ldots \forall x_m.\ \phi$ is not provable.

Exercise 6 : Provably total recursive functions

Let f be a total recursive function from $\mathbb{N}$ to $\mathbb{N}$, and $A[x, y]$ a formula representing f. As f is total, $\mathbb{N} \models \forall x \exists y.\ A[x, y]$. We say f is provably total if $\forall x \exists y.\ A[x, y]$ is provable in PA. The goal of this exercise is to prove that there are total does not imply provably total.

1. Let ϕ be a formula of free variables $x_1, \ldots, x_k$. Show that the set $\{(n_1, \ldots, n_k) \in \mathbb{N}^k :\ \vdash \phi[\underline{n_1}, \ldots, \underline{n_k}]\}$ is recursively enumerable.
2. Show that a total function $f : \mathbb{N} \to \mathbb{N}$ is recursive iff there is a formula representing f.
3. Show that there is a partial recursive function h with two arguments such that for all $n \in \mathbb{N}$:
   - If $\phi[x, y]$ is formula with free variables x, y and there exists $m \in \mathbb{N}$ such that $\phi[\underline{m}, \underline{n}]$ is provable in PA, then $\phi\left[\underline{h(\ulcorner \phi \urcorner, n)}, \underline{n}\right]$ is also provable.
   - If $\phi[x, y]$ is formula with free variables x, y and there is no $m \in \mathbb{N}$ such that $\phi[\underline{m}, \underline{n}]$ is provable in PA, then $h(\ulcorner \phi \urcorner, n)$ is not defined.
   - else $h(a, n) = 0$.

   You can use that the function $a, m, n \mapsto \ulcorner \phi[\underline{m}, \underline{n}] \urcorner$ if $\ulcorner \phi \urcorner = a$ where ϕ has two free variables x, y, and else 0 is recursive.
4. We define $g : \mathbb{N}^3 \to \mathbb{N}$ in the following way:
   - if $a = \ulcorner \phi \urcorner$ where $\phi[x, y]$ has two free variables x, y and $b = \ulcorner \pi \urcorner$ where π is a proof in PA of $\forall x \exists y.\ \phi(x, y)$, then $g(a, b, n) = h(a, n)$.

- else, $g(a, b, n) = 0$.

Show that g is a total recursive function. You can use that recursive functions can be defined by cases.

5. Show that there exists total recursive functions that are not provably total.