# Homework 2: Constraint system solving

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The goal of this homework is to propose an algorithm to find a solution (if there is any) to a given deducibility constraint system. First, we want a class of constraint system where it is "easy" to show that they have a solution.

**Definition 1:** Solved constraint system

A constraint system $\mathcal{C}$ is said to be *solved* if it is in the form

$$\mathcal{C} = \bigwedge_{i=1}^{n} (T_i \vdash^? x_i)$$

where, for all $i \in [\![1; n]\!]$, $x_i$ is a variable in $\mathcal{X}$.

**Question 1**

*Show that any solved constraint system have a solution.*

**Definition 2:** Simplification rules for constraint system

We will consider a set of simplification rules for constraint system:

$$\mathcal{C} \wedge \left(T \vdash^? u\right) \rightsquigarrow \mathcal{C} \quad \text{if } T \cup \left\{x \in \mathcal{X} \mid (T' \vdash^? x) \in \mathcal{C}, T' \subseteq T\right\} \vdash u \qquad (R_1)$$

$$\mathcal{C} \wedge \left(T \vdash^? u\right) \rightsquigarrow_\sigma \mathcal{C}\sigma \wedge \left(T\sigma \vdash^? u\sigma\right) \quad \text{if } t \in \mathsf{st}(T), \sigma = \mathsf{mgu}(t, u), t \neq u \text{ and } t, u \notin \mathcal{X} \qquad (R_2)$$

$$\mathcal{C} \wedge \left(T \vdash^? u\right) \rightsquigarrow_\sigma \mathcal{C}\sigma \wedge \left(T\sigma \vdash^? u\sigma\right) \quad \text{if } t, v \in \mathsf{st}(T), \sigma = \mathsf{mgu}(t, v), t \neq v \qquad (R_3)$$

$$\mathcal{C} \wedge \left(T \vdash^? u\right) \rightsquigarrow \bot \quad \text{if } \mathsf{fv}(T \cup \{u\}) = \emptyset \text{ and } T \not\vdash u \qquad (R_4)$$

$$\mathcal{C} \wedge \left(T \vdash^? f(u_1, \dots, u_n)\right) \rightsquigarrow \mathcal{C} \wedge \bigwedge_{i=1}^{n} \left(T \vdash^? u_i\right) \quad \text{if } (f/n) \in \Sigma \text{ is a constructor symbol} \qquad (R_f)$$

**Question 2 (Correctness)**

Let $\mathcal{C}$ be a constraint system. Suppose that $\mathcal{C} \rightsquigarrow_\sigma \mathcal{C}'$. *Show that $\mathcal{C}'$ is also a constraint system.*

**Question 3 (Termination)**

*Show that there is not infinite sequence $\left(\mathcal{C}_n\right)_{n \in \mathbb{N}}$ such that*

$$\forall n \in \mathbb{N},\ \mathcal{C}_n \rightsquigarrow_{\sigma_n} \mathcal{C}_{n+1}.$$

**Hint:** Use a lexicographical order on $(v, s)$ where $v$ is the number of variables and $s$ is the size of the constraint system, for some good notion of size to be defined.

**Question 4 (Soundness)**

In this question, we will show the following theorem for completeness of the constraint system simplification algorithm.

***Theorem 1:*** *Completeness of the constraint system simplification*

*Let $\mathcal{C}$ be an unsolved deducibility constraint system and let $\theta$ be a solution of $\mathcal{C}$. Then, there is a deducibility constraint system $\mathcal{C}'$, a substitution $\sigma$, and a solution $\theta'$ of $\mathcal{C}'$ such that*

$$\mathcal{C} \rightsquigarrow_\sigma^* \mathcal{C}' \qquad \text{and} \qquad \theta = \sigma\theta'.$$

First, we define the notion of *simple proofs*.

**Definition 3:** *Left minimal proofs, Simple proofs*

*Let $(T_i)_{i=1}^n$ be an increasing sequence of sets of terms,* i.e., *for all $i \in [\![1; n-1]\!]$, $T_i \subseteq T_{i+1}$.*
*We say that a proof $\Pi$ of $T_i \vdash u$ is* left minimal *when, if there is a proof of $T_j \vdash u$ for some $1 \leqslant j < i$, then $\Pi$ is also a proof of $T_j \vdash u$.*
*Moreover, a proof $\Pi$ is said to be* simple *when all its subproofs are left minimal and there is no repeated label on any branch.*

1. Let $(T_i)_{i=1}^n$ be an increasing sequence of sets of terms, *i.e.*, for all $i \in [\![1; n-1]\!]$, $T_i \subseteq T_{i+1}$. Let $u$ be a term such that $T_i \vdash u$. *Show that there exists a simple proof $\Pi$ of $T_i \vdash u$.*

2. Let $\mathcal{C}$ be an unsolved constraint system. Let $\theta$ be a solution of $\mathcal{C}$ and let $T_i \vdash^? u_i$ be a minimal unsolved constraint system of $\mathcal{C}$. Let $u$ be a term. We suppose that there is a simple proof of $T_i\theta \vdash u$ having as last rule an axiom or a decomposition. *Show that there exists a term $t \in \mathsf{st}(T_i) \setminus \mathcal{X}$ such that $t\theta = u$.*

3. Let $n \in \mathbb{N}$ be a natural number and let $\mathcal{C} = \left\{T_i \vdash^? x_i\right\}_{i=0}^n$ be a constraint system. Let $\sigma$ be a solution of $\mathcal{C}$. We suppose

   (a) $T_n$ does not contain two distinct subterms $t_1, t_2 \in \mathsf{st}(T_n)$ such that $t_1 \neq t_2$ and $t_1\sigma = t_2\sigma$ ;

   (b) $u$ is a non-variable subterm of $T_n$.

   *Show that $T'_n \vdash u$ where* $T'_n \stackrel{def}{=} T_n \cup \left\{x \in \mathcal{X} \;\middle|\; (T \vdash^? x) \in \mathcal{C} \wedge T \subsetneq T_n\right\}$.

4. *Show the completeness theorem for only one step (*i.e. $\mathcal{C} \rightsquigarrow_\sigma \mathcal{C}'$*). Conclude.*