# Langages Formels

# TD 7 - Révisions

Isa Vialard
vialard@lsv.fr

March 15, 2024

**Exercise 1 : Intermède: congruences**

Let's explore the fabulous world of congruences!

1. Using congruences, prove that $\{ xcx \ : \ x \in \{ a, b \}^* \}$ is not regular.

   There is a different class for each $a^n$ in the syntaxic congruence.

2. Same question for any infinite subset of $\{ a^n b^n \ : \ n \in \mathbb{N} \}$.

   Idem.

3. Consider the regular language $L$ represented by $a^*b^* + b^*a^*$.
   (a) Draw the minimal automaton for $L$.
   (b) Give a regular expression describing each of the equivalence classes of the syntactic congruence of $L$, denoted $\equiv_L$.

   $a^+b^+$, $a^+$, $b^+$, $\varepsilon$ and $\Sigma^* \setminus L$.

4. Let $\Sigma$ be an alphabet. Let $\equiv$ be a a congruence of finite index over $\Sigma^*$. Prove that any equivalence class of $\equiv$ is a regular language of $\Sigma^*$.

   Let $L$ be an equivalence class of $\equiv$. Then $\equiv$ saturates $L$ so $L$ is saturated by a finite congruence therefore $L$ is regular.

**Exercise 2 : Two way automata (Boustrophédon)**

A two way automaton is a finite automaton which, for each transition, can move its reading head one step to the right or one step to the left. Equivalently, it is a Turing machine with one ribbon which cannot write.

1. Build a two way automaton with $O(n)$ states that accept $\Sigma^* a \Sigma^n$.
2. Show that all language accepted by a deterministic two way automaton is regular.
3. Show that from any deterministic two way automaton with $n$ states, we can construct an equivalent deterministic finite automaton with $2^{O(n^2)}$ states.

Réponse rédigée en détail ici:

https://www.cs.cornell.edu/courses/cs682/2008sp/Handouts/2DFA.pdf

L'automate qu'ils construisent est en $O(n^n)$. Les états sont les fonctions $Tx : Q+1 \to Q+1$. Idée derrière $Tx$ : pour n'importe quel mot $xz$, si la tête de lecture revient dans $x$ à un état $q$, elle ressort à un état $p$ (qui ne dépend pas de $z$), et alors on choisit $Tx(p) = q$. Comme il y a un nombre fini de fonctions différentes de $Q+1 \to Q+1$, on obtient une congruence $x \cong y$ ssi $Tx = Ty$.

**Exercise 3 : Selection property**

A morphism $\mu : A^* \to B^*$ has the *selection property* iff for every regular language $L$, there exists a regular language $K \subseteq L$ such that $\mu$ is injective over $K$ and $\mu(K) = \mu(L)$. The goal of this exercise is to show that every morphism has the selection property.

1. Show that all injective morphisms have the selection property.
2. Show that if morphisms $\mu$ and $\nu$ have the selection property, then the morphism $\mu \circ \nu$ also has it.

For all regular language $L$, $\mu$ has the selection property so there exists $K \subseteq \nu(L)$ regular such that $\mu$ is injective on $K$ and $\mu(K) = \mu \circ \nu(L)$. Take $L' = \nu^{-1}(K)$. $L'$ is regular because regularity is closed by inverse morphism. $\nu$ has the selection property so there exists $K' \subseteq L' \subseteq L$ regular such that $\nu$ is injective on $K'$ and $\nu(K') = \nu(L') = K$, hence $\mu \circ \nu(K') = \mu \circ \nu(L)$ and $\mu \circ \nu$ is injective on $K'$.

We call *projection* a morphism $\pi : A^* \to B^*$ such that for every letter $a \in A$, $\pi(a) = a$ or $\pi(a) = \varepsilon$.

3. Show that for every morphism $\mu : A^* \to B^*$, there exists an alphabet $C$, an injective morphism $\iota : A^* \to C^*$ and a projection $\pi : C^* \to B^*$ such that $\mu = \pi \circ \iota$.

Idée : on rajoute des lettres à $A$ pour différencier les mots qui collisionnent par $\mu$, puis on efface ces lettres avec $\pi$.

We call *elementary projection* a projection $\pi : A^* \to B^*$ such that there exists a unique letter $a \in A$ such that $\pi(a) = \varepsilon$.

4. Show that every projection is the composition of elementary projections.
5. Show that all elementary projection has the selection property. *(Cette question est plus dure qu'il n'y parait.)*

Je n'ai pas la solution (même si je suis sûr que c'est vrai), si vous trouvez envoyez-moi un mail (j'offre un 20 en contrôle continu à la première personne qui trouve).

6. Conclude.