Laboratoire Méthodes Formelles

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école normale supérieure paris—saclay

# Beyond Decisiveness: When Statistical Verification Meets Numerical Verification

Benoît Barbot (LACL), Patricia Bouyer (LMF), Serge Haddad (LMF)

Supported by ANR projects MAVeriQ and BisoUS
(not submitted yet, hopefully soon on ArXiV)

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Design algorithms to estimate probabilities in some **infinite-state** Markov chains, **with guarantees**

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Design algorithms to estimate probabilities in some **infinite-state** Markov chains, **with guarantees**

Our contributions

- Review two existing approaches (approximation algorithm and estimation algorithm) and specify the required hypothesis for correctness
- Propose an approach based on **importance sampling** and **abstraction** to partly relax the hypothesis
- Analyze empirically the approaches

# Discrete-time Markov chains

Discrete-time Markov chain (DTMC)

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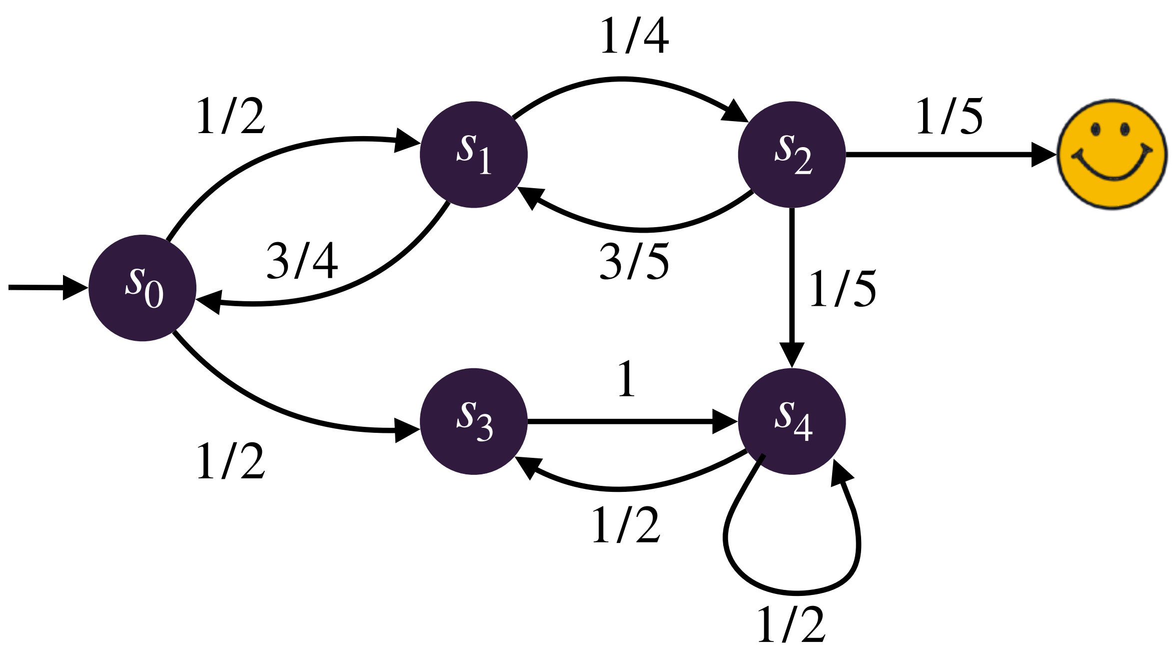


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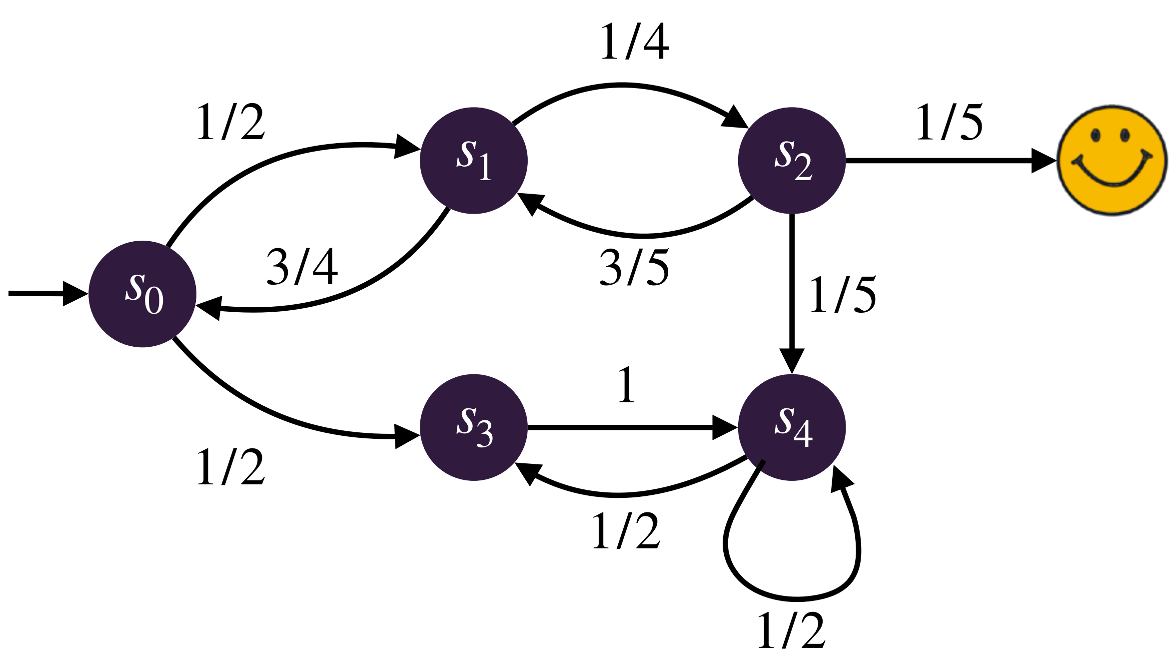


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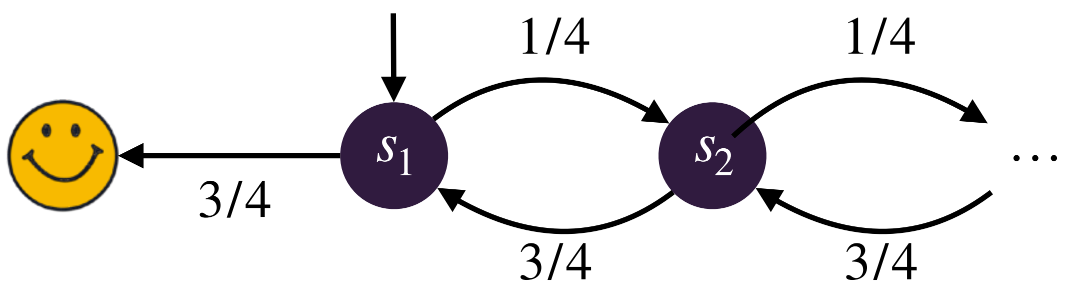


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(random walk of parameter $1/4$)

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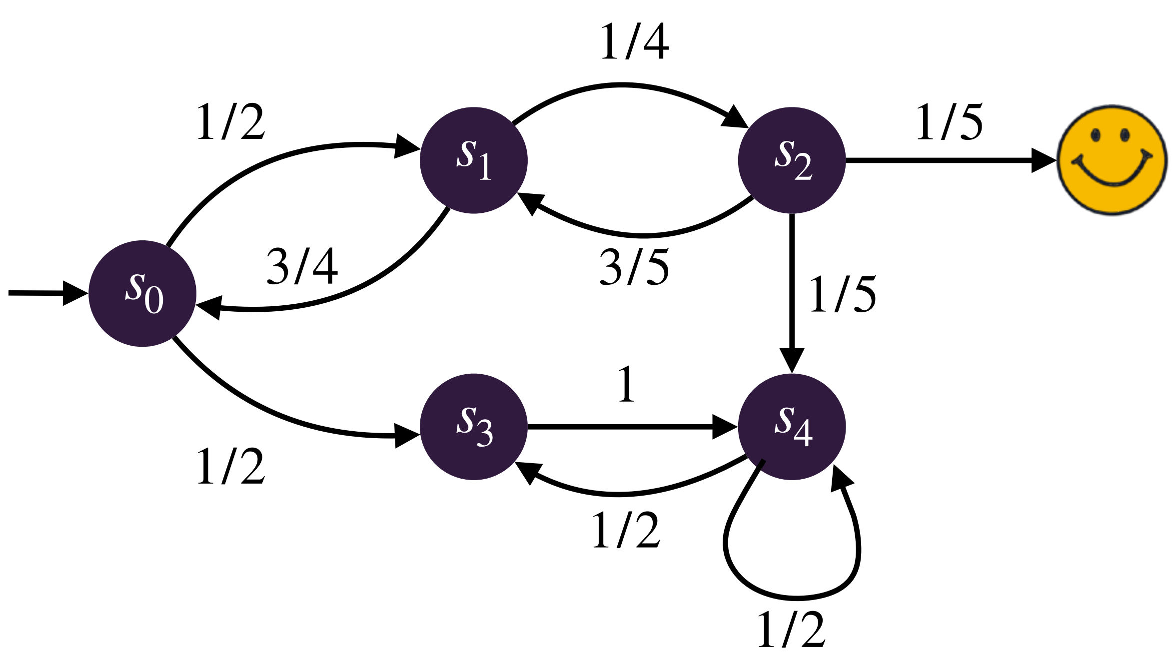


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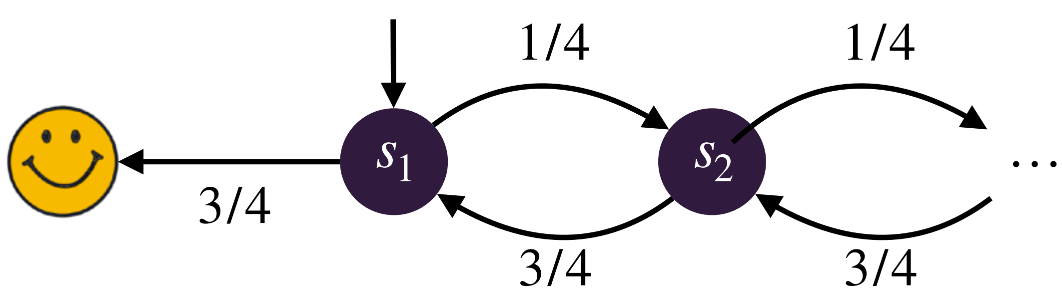


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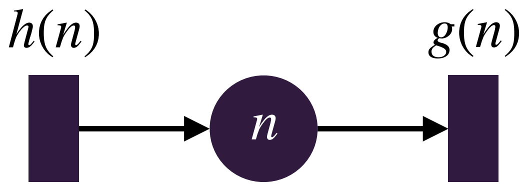
# High-level models for (infinite) Markov chains

- Queues



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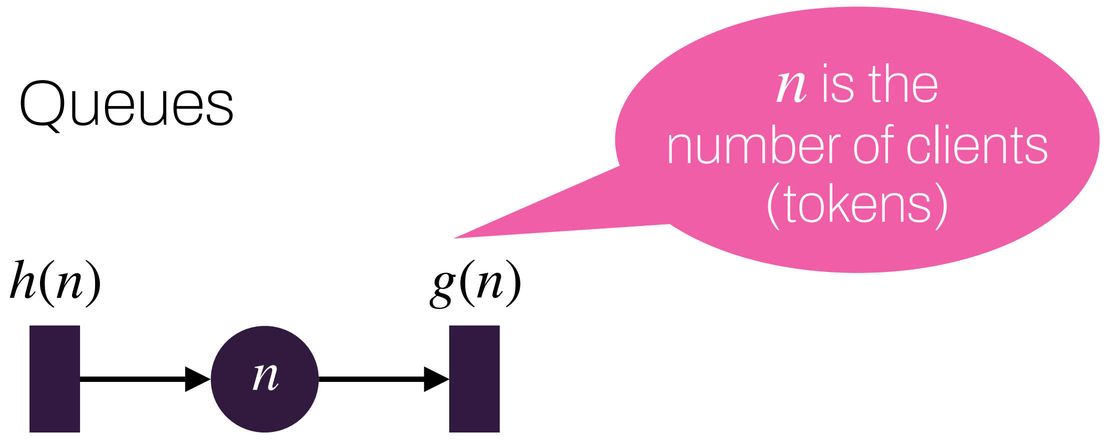

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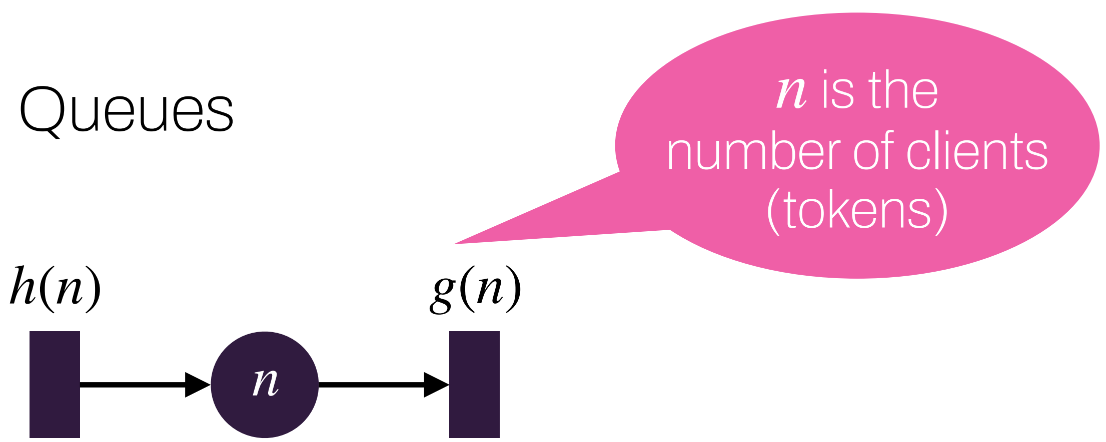


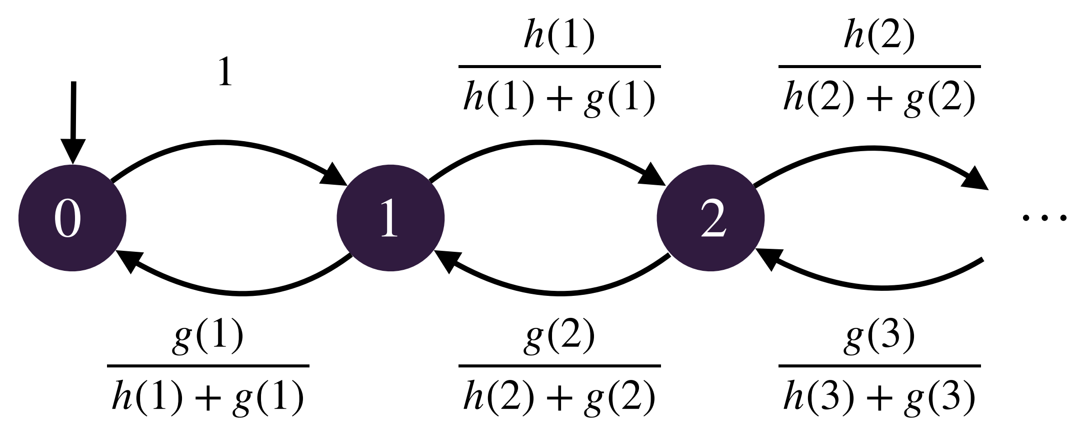

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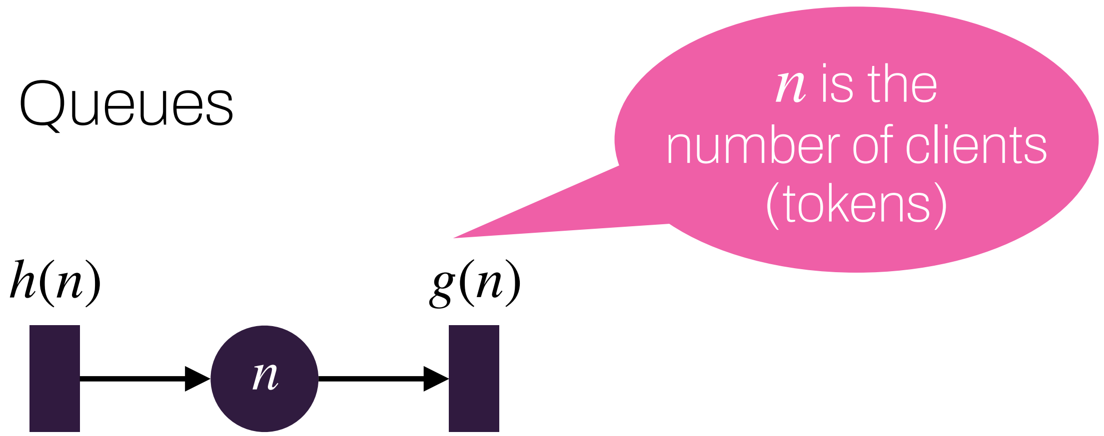


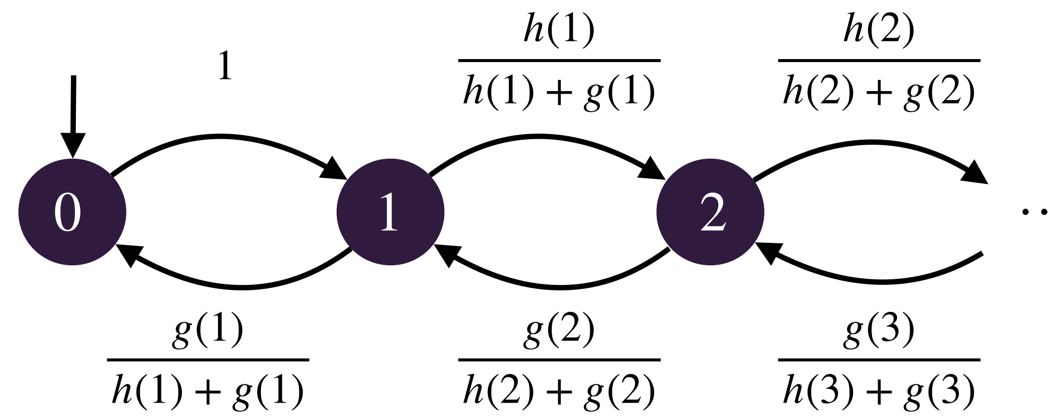


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$$A \xrightarrow{1} C \quad A \xrightarrow{n} BB \quad B \xrightarrow{5} \varepsilon$$

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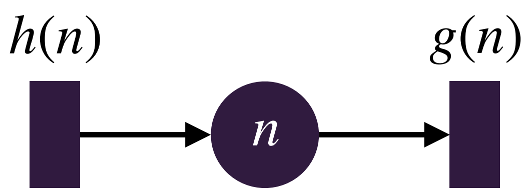


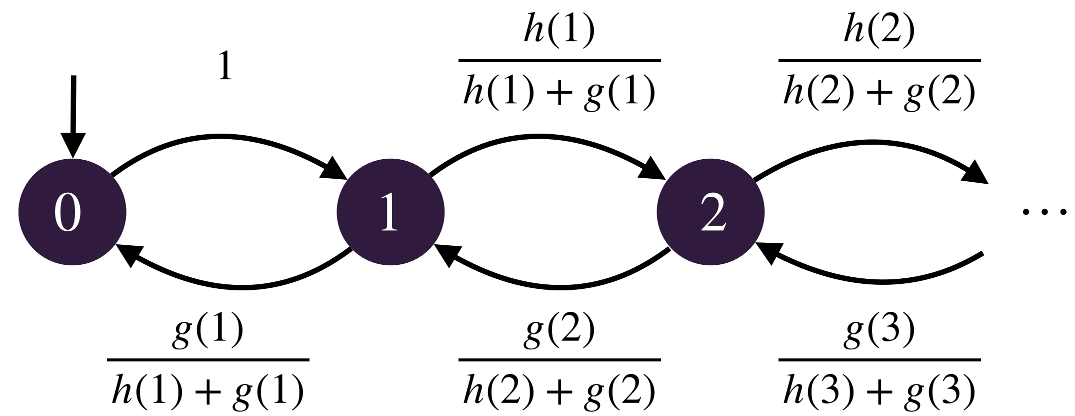


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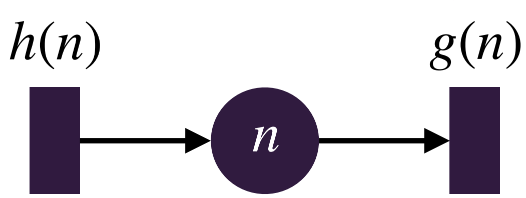


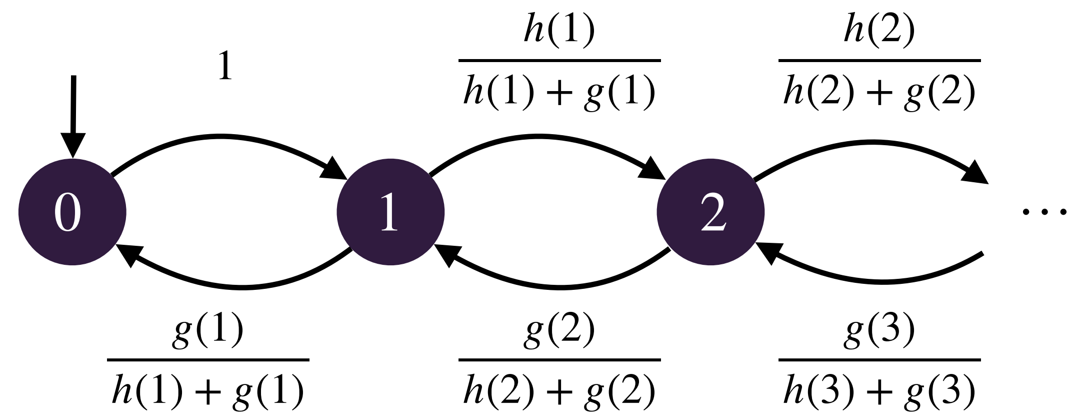


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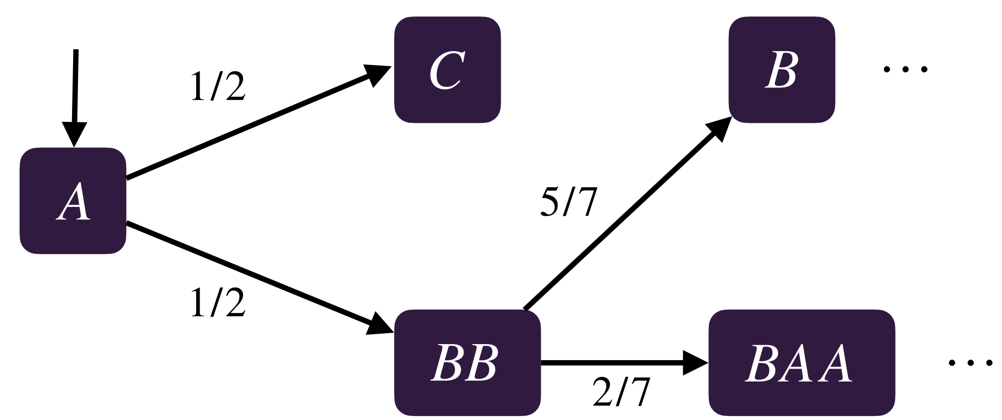

# Quantitative analysis of Markov chains

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Very useful even beyond reachability properties (decomposition in BSCCs)

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- Specific approaches for **decisive** Markov chains

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## Decisiveness

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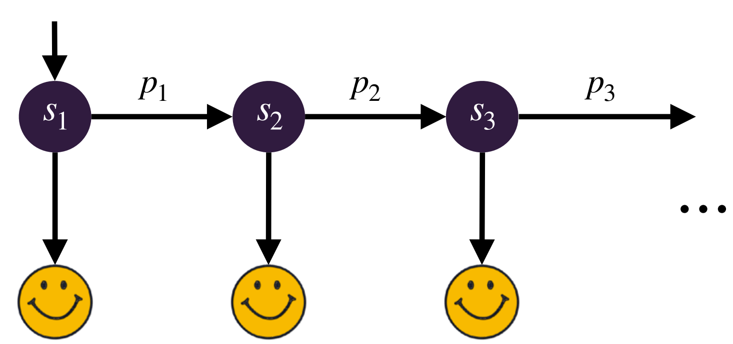


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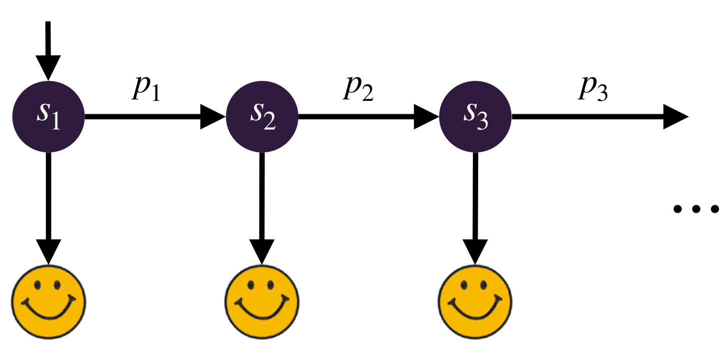


- $\mathbb{P}(\mathbf{G}\neg 🙂) = \prod_{i \geq 1} p_i$
- Decisive iff this product equals $0$

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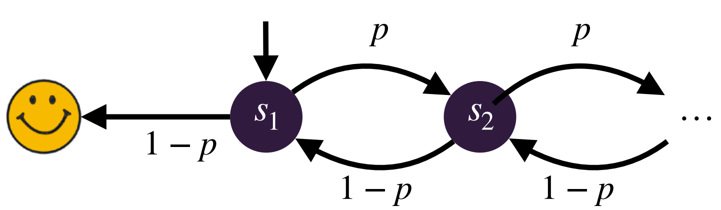


- Recurrent random walk ($p \leq 1/2$): decisive
- Transient random walk ($p > 1/2$): not decisive

[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)

# Deciding decisiveness?

**Classes where decisiveness can be decided**

- Probabilistic pushdown automata with constant weights [ABM07]
- Random walks with polynomial weights [FHY23]
- So-called probabilistic homogeneous one-counter machines with polynomial weights (this extends the model of quasi-birth death processes) [FHY23]

*[ABM07] P.A. Abdulla, N. Ben Henda, R. Mayr. Decisive Markov chains (LMCS, 2007)*
*[FHY23] A. Finkel, S. Haddad, L. Yé. About decisiveness of dynamic probabilistic models (CONCUR'23)*

# Approximation scheme

- Aim: compute probability of $\mathbf{F}$

[IN97] P. Iyer, M. Narasimha. Probabilistic lossy channel systems (TAPSOFT'97)

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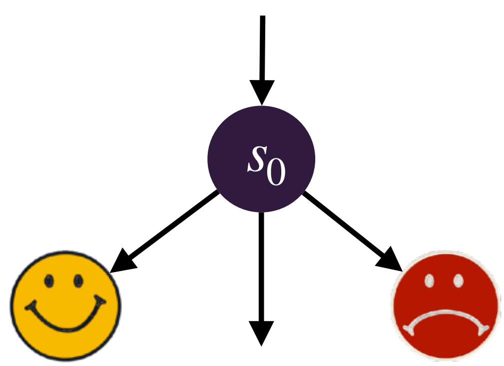


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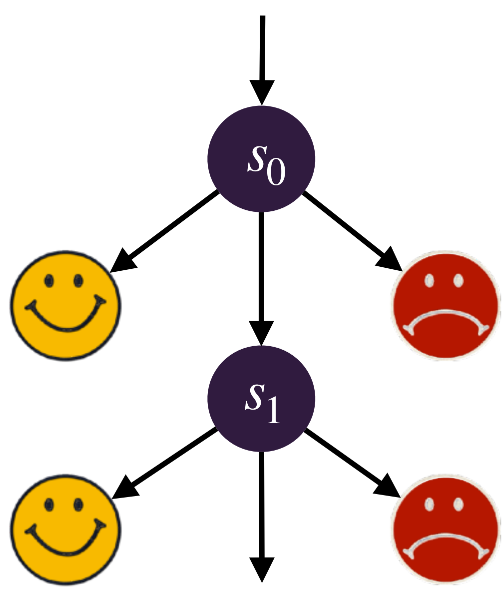


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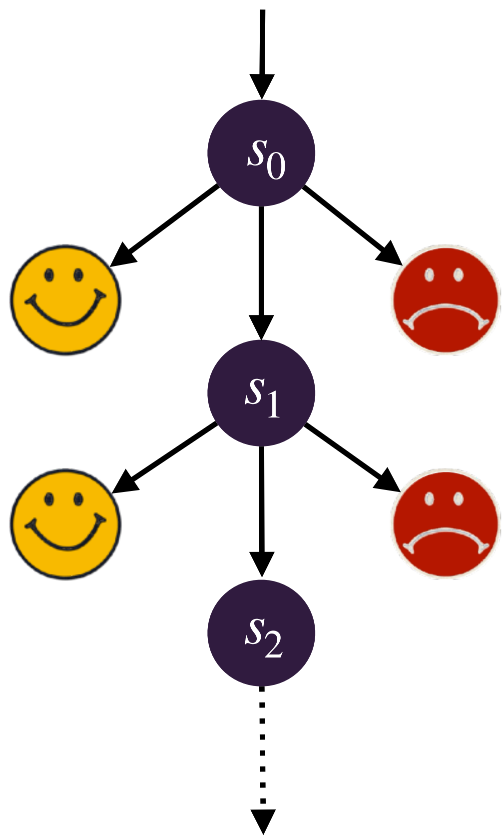


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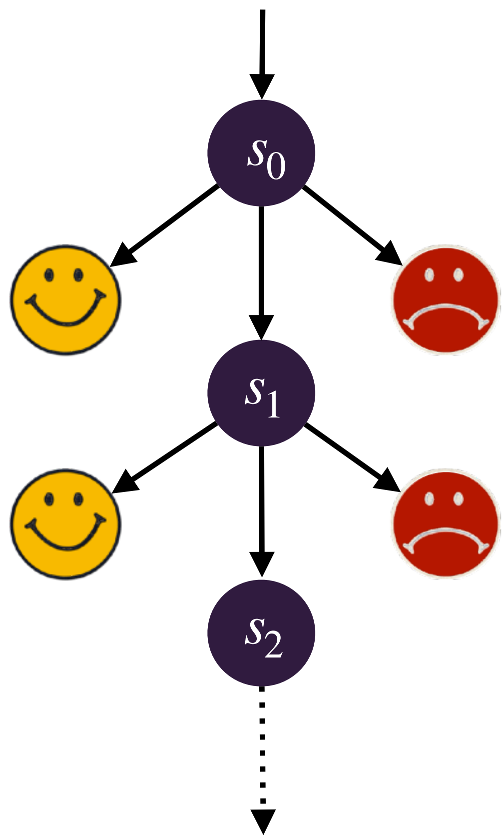


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$$\begin{cases} p_n^{\text{yes}} = \mathbb{P}(\mathbf{F}_{\leq n} \text{🙂}) \\ p_n^{\text{no}} = \mathbb{P}(\neg \text{🙂} \, \mathbf{U}_{\leq n} \text{☹}) \end{cases}$$

until $p_n^{\text{yes}} + p_n^{\text{no}} \geq 1 - \varepsilon$

$$p_1^{\text{yes}} \leq \mathbb{P}(\mathbf{F} \text{🙂}) \leq 1 - p_1^{\text{no}}$$

$$\text{IV} \qquad\qquad\qquad \text{VI}$$

$$p_2^{\text{yes}} \leq \mathbb{P}(\mathbf{F} \text{🙂}) \leq 1 - p_2^{\text{no}}$$

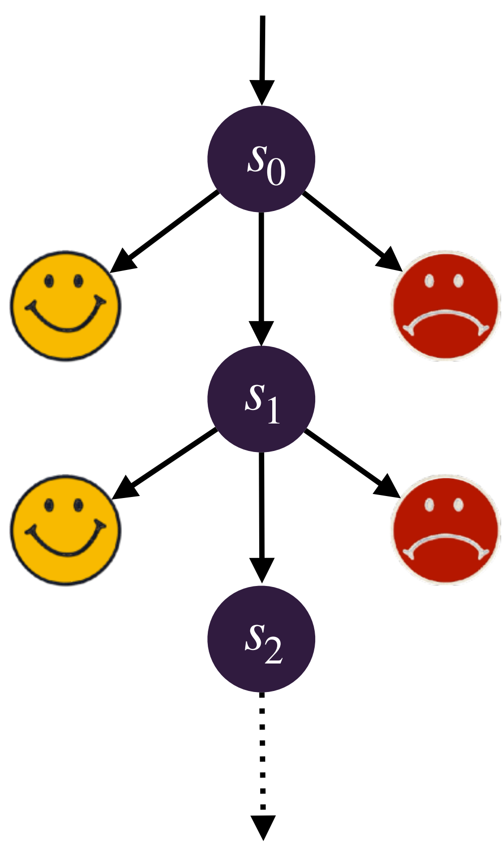
$$\text{IV} \qquad \vdots \qquad \text{VI}$$

Does it converge?

*[IN97] P. Iyer, M. Narasimha. Probabilistic lossy channel systems (TAPSOFT'97)*

# Approximation scheme

- Aim: compute probability of $\mathbf{F}$ ☺
- ☹ $= \{s \in S \mid s \not\models \exists\mathbf{F}$ ☺ $\}$

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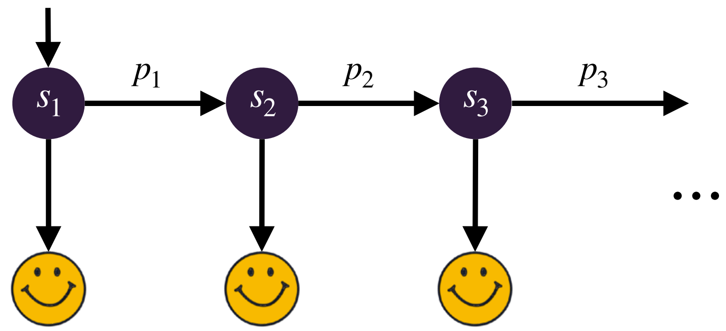
# Non-converging example




with $\prod_{i \geq 1} p_i > 0$

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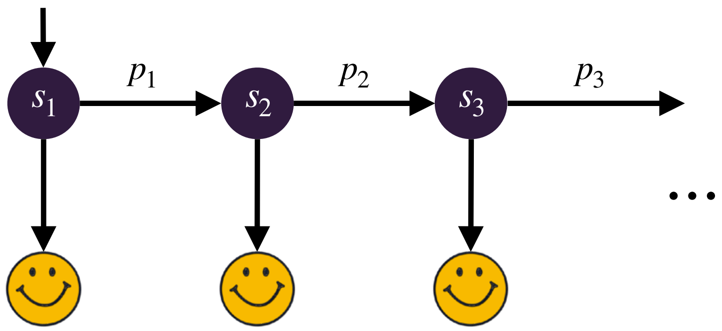


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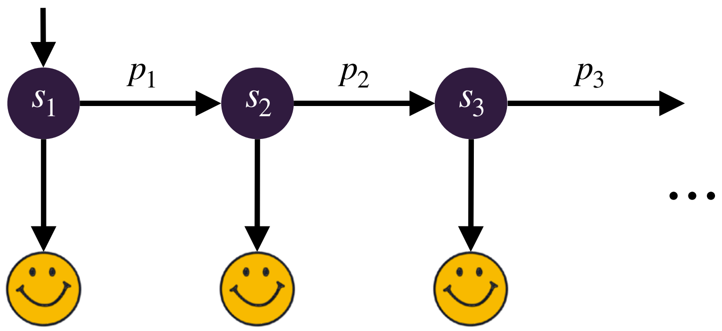


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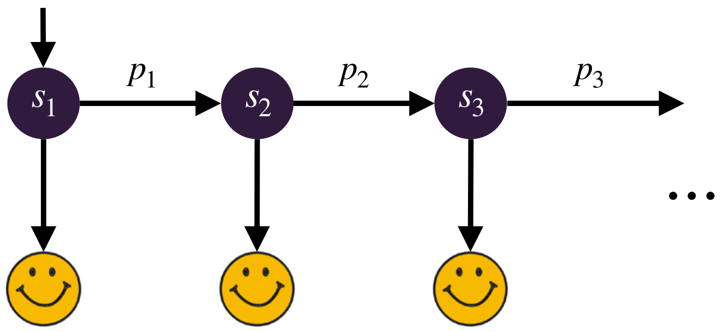


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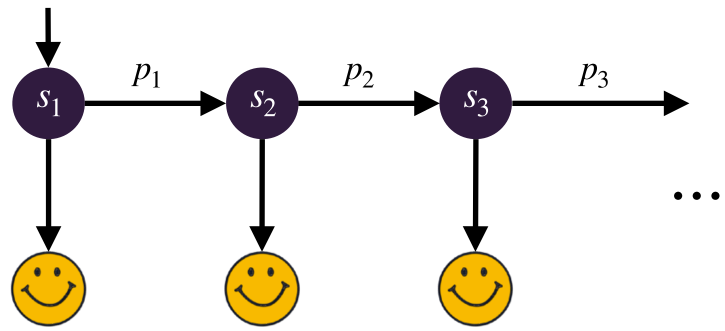


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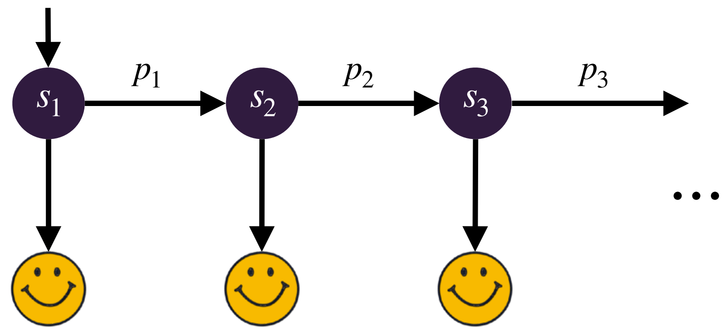


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The approximation scheme does not converge

# Termination of the approx. scheme

**Approximation scheme**

Given $\varepsilon > 0$:

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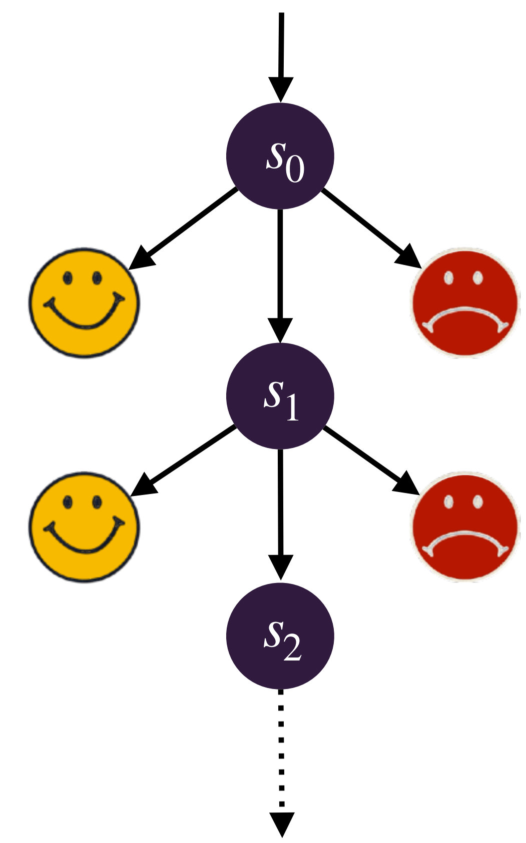


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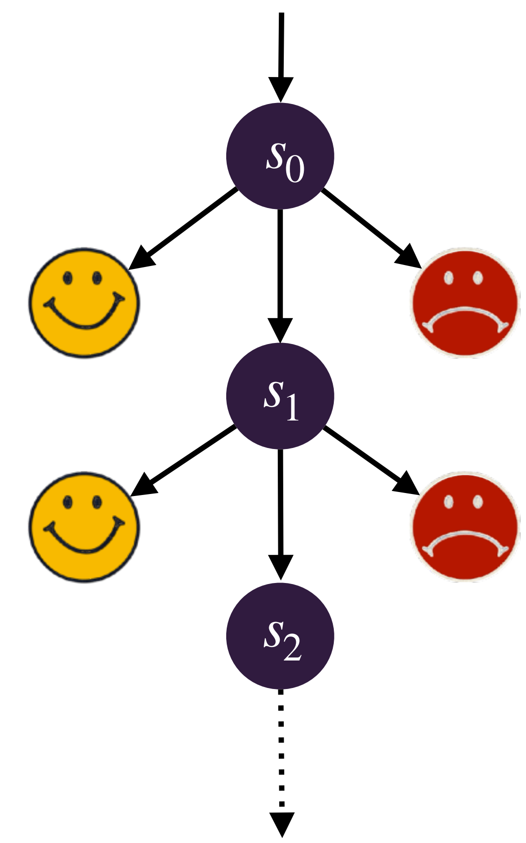


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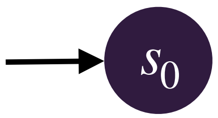
# Statistical model-checking

Sample $N$ paths



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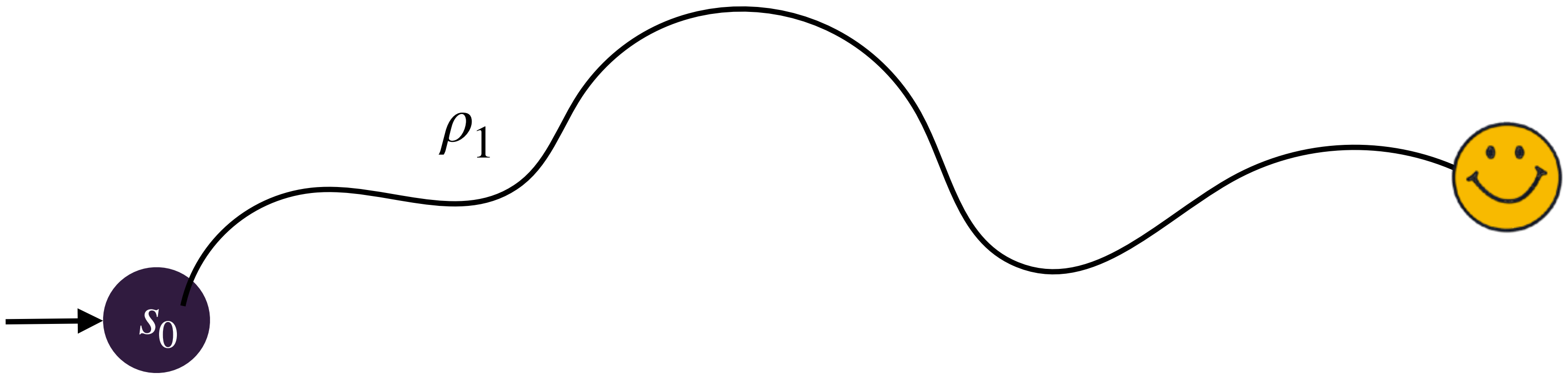


$n_1 = 1$

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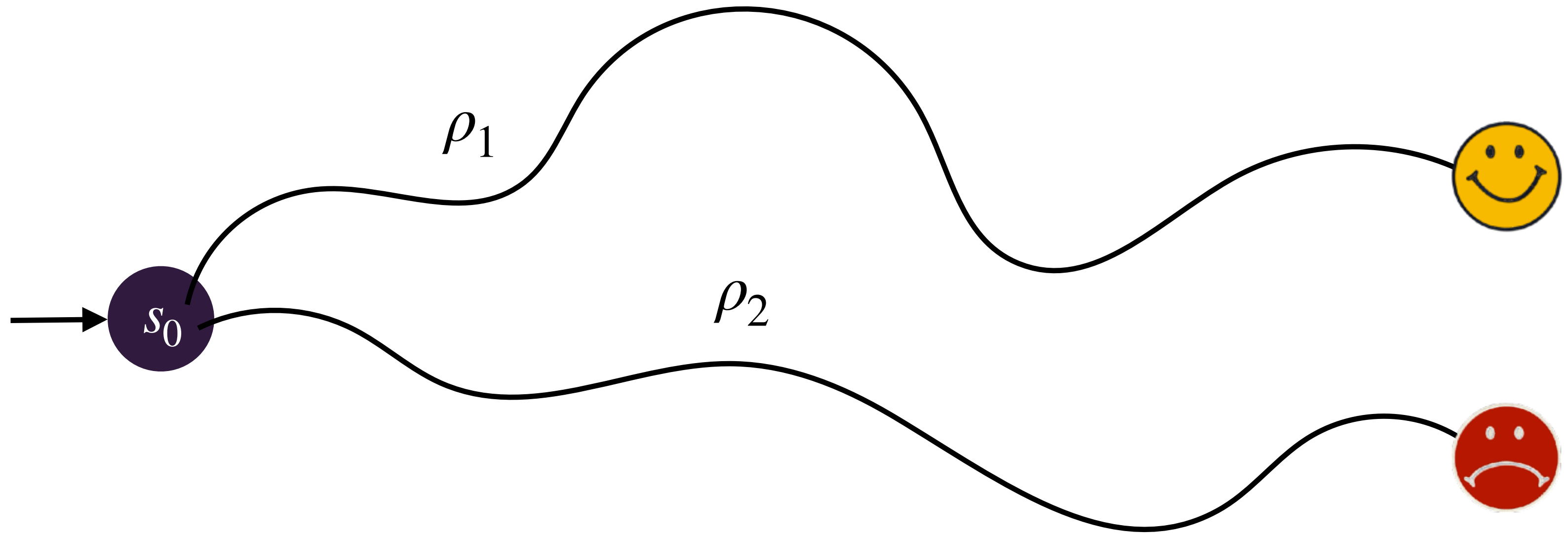


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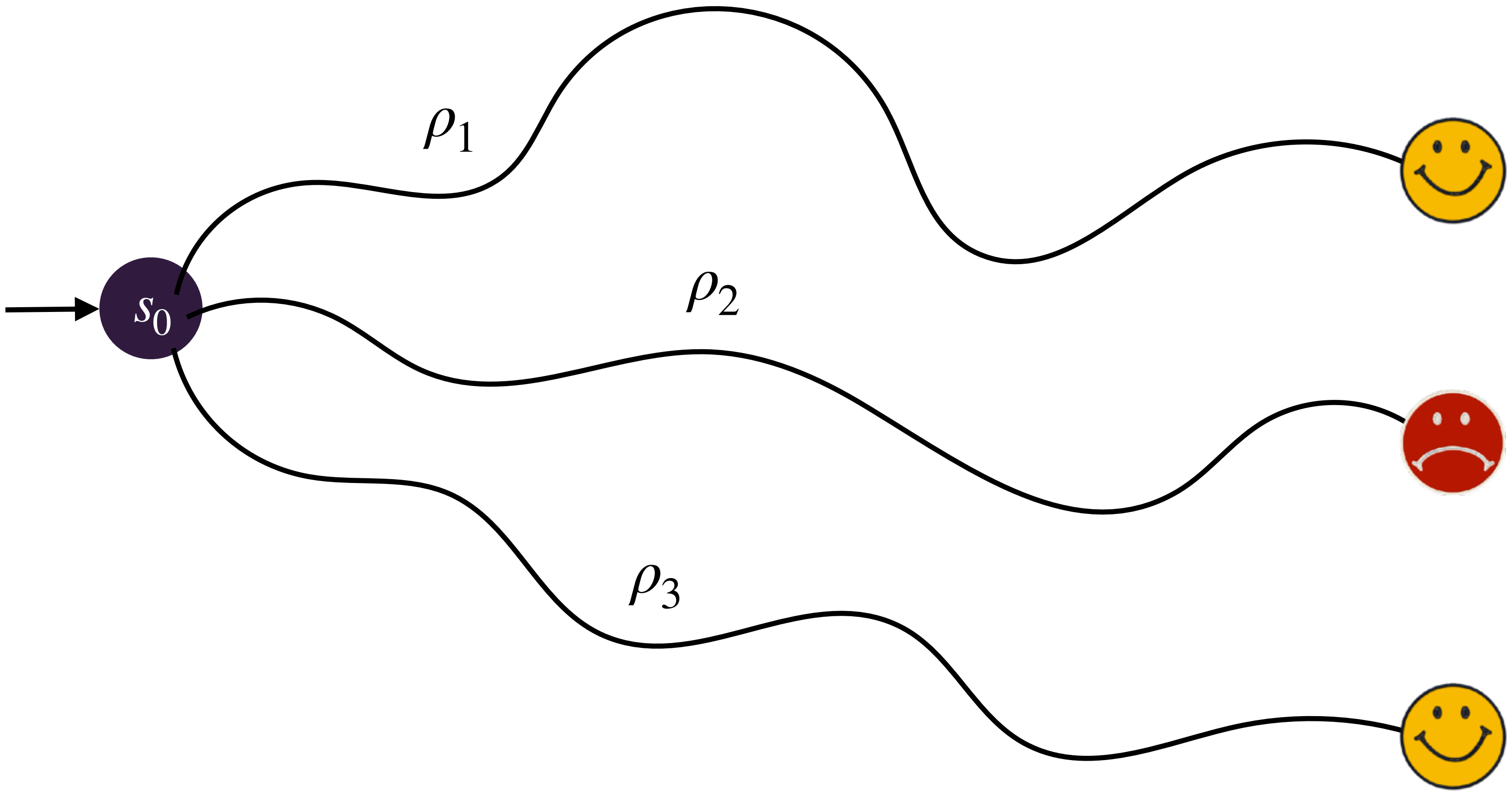


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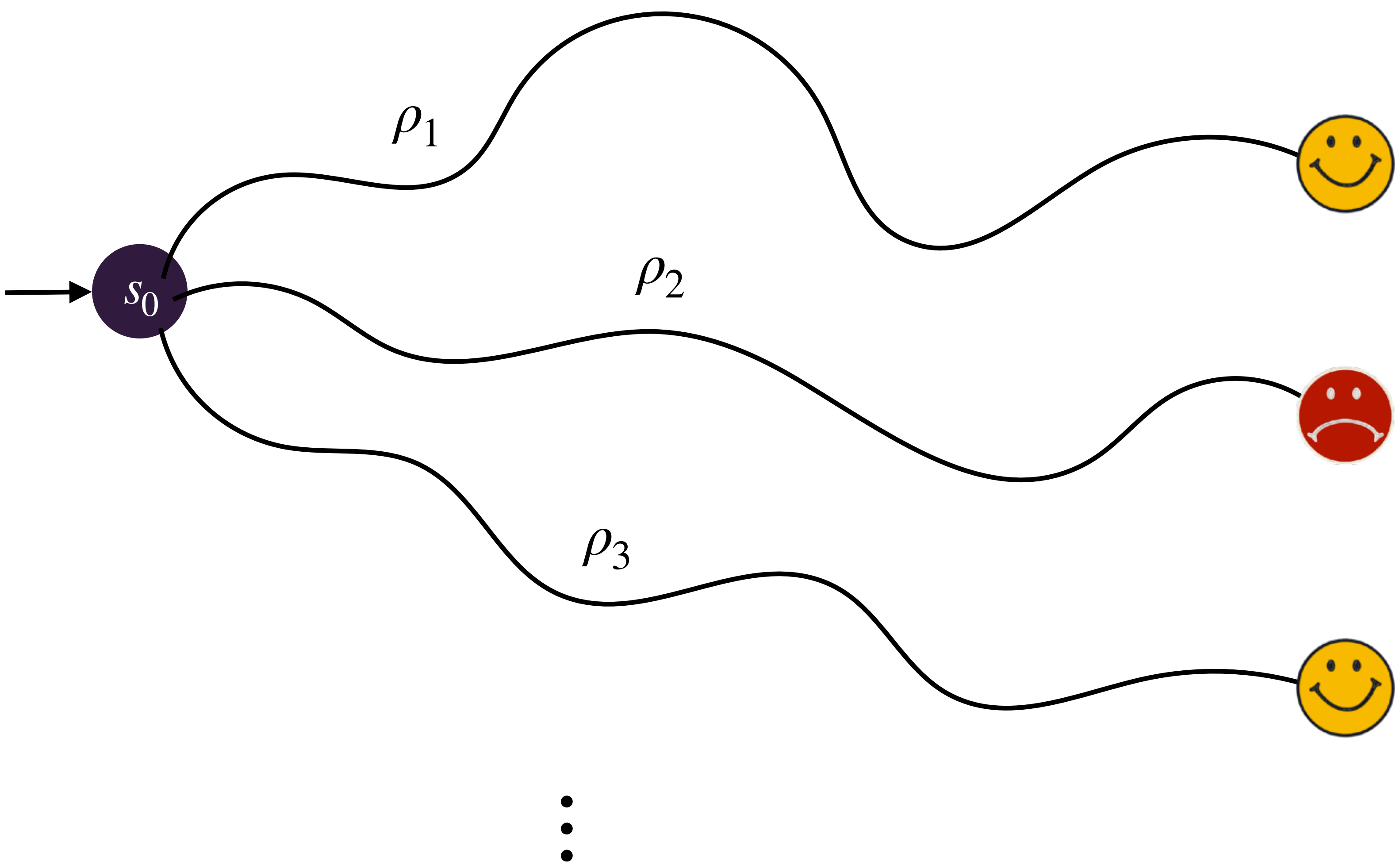


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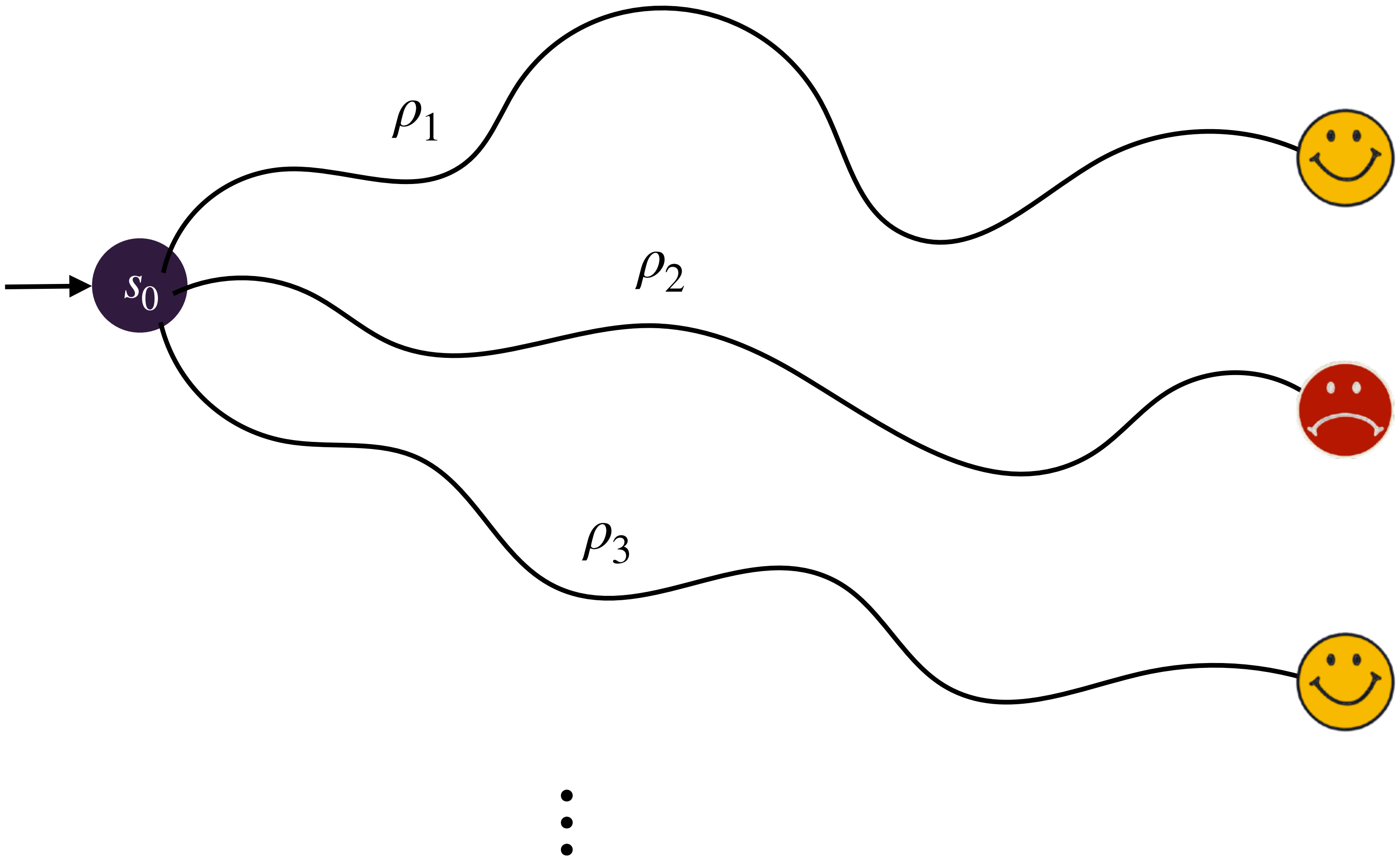


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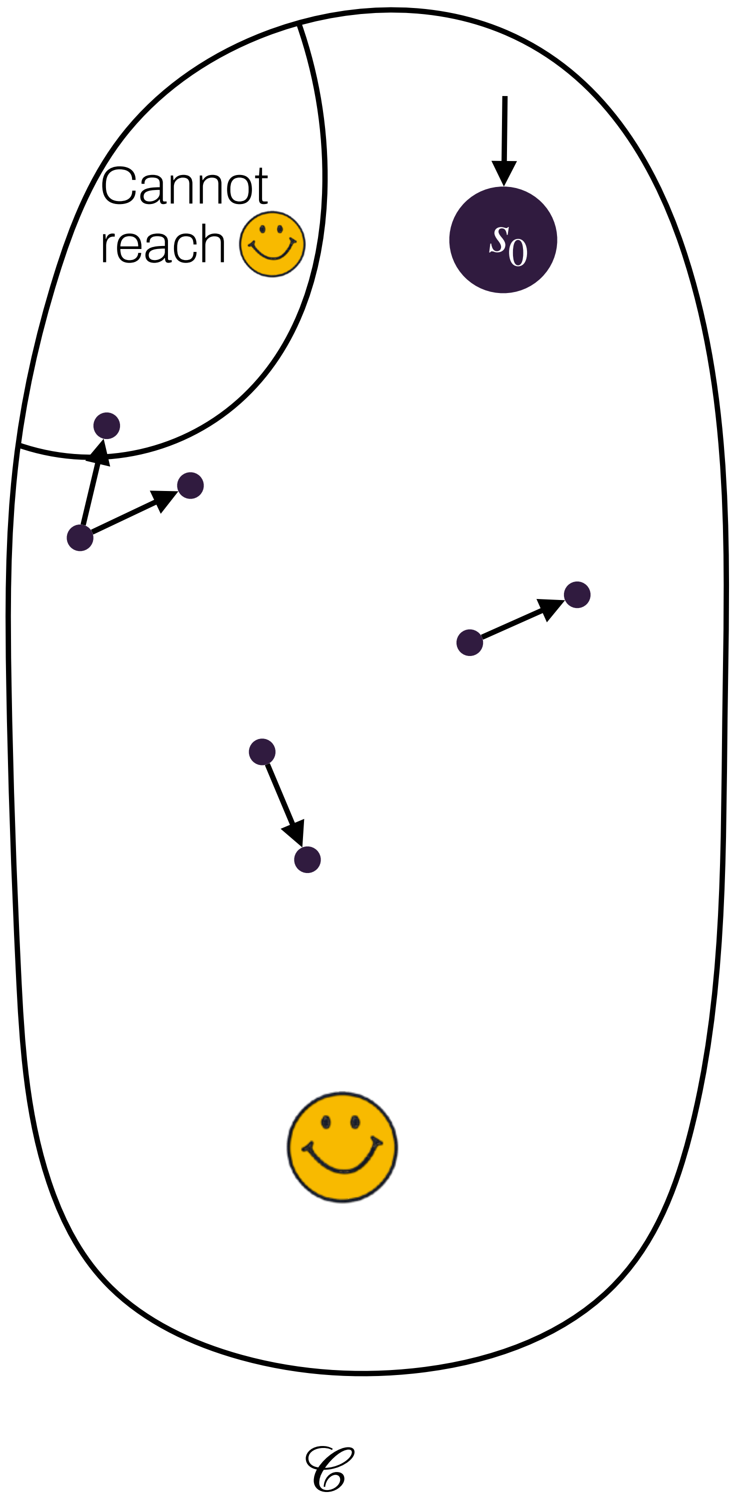
Return $\dfrac{n_N}{N}$ + some confidence interval

# Termination and efficiency

**Termination** (To our knowledge, never expressed like this)

$\mathscr{C}$ is decisive from $s_0$ w.r.t. 
iff
a sampled path starting at $s_0$ almost-surely hits  or 

# Decisiveness vs recurrence



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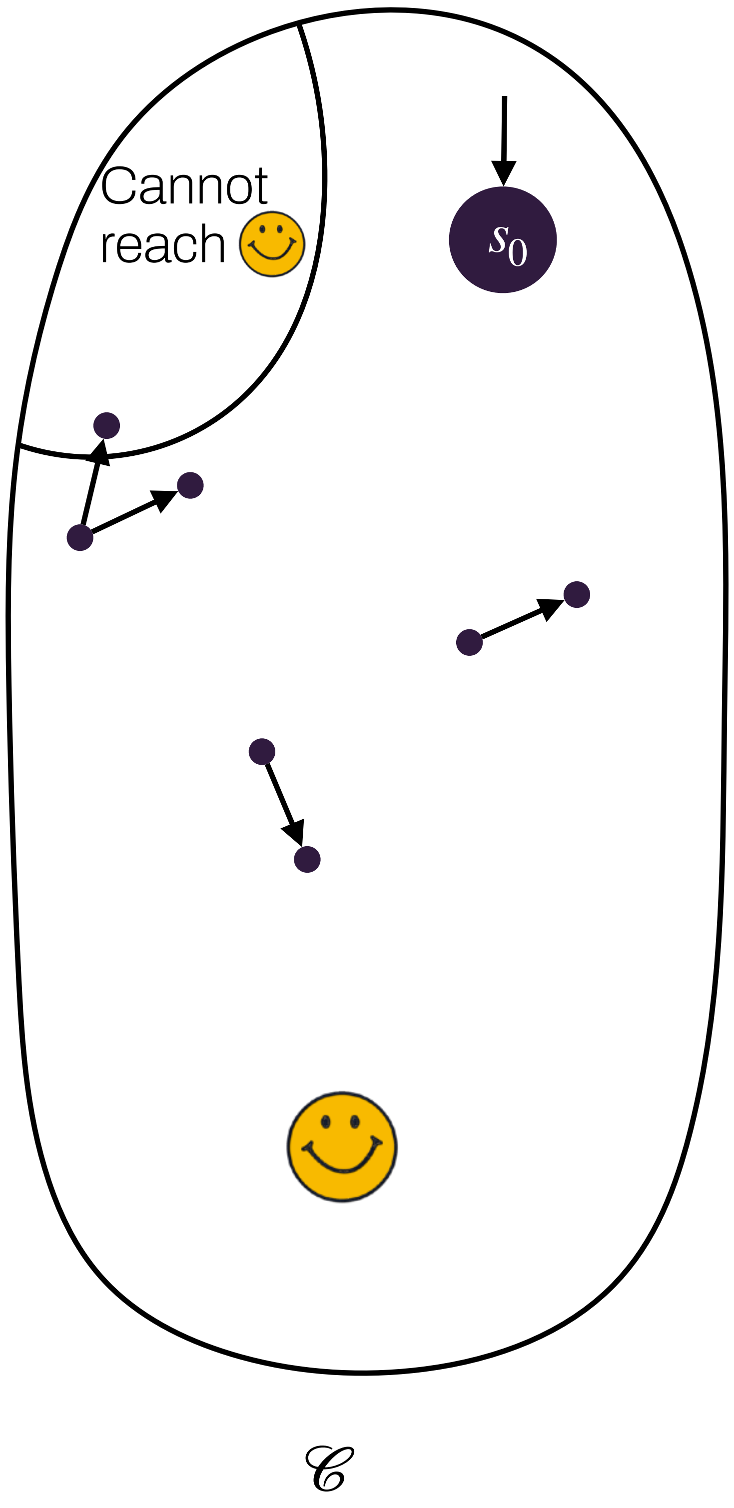


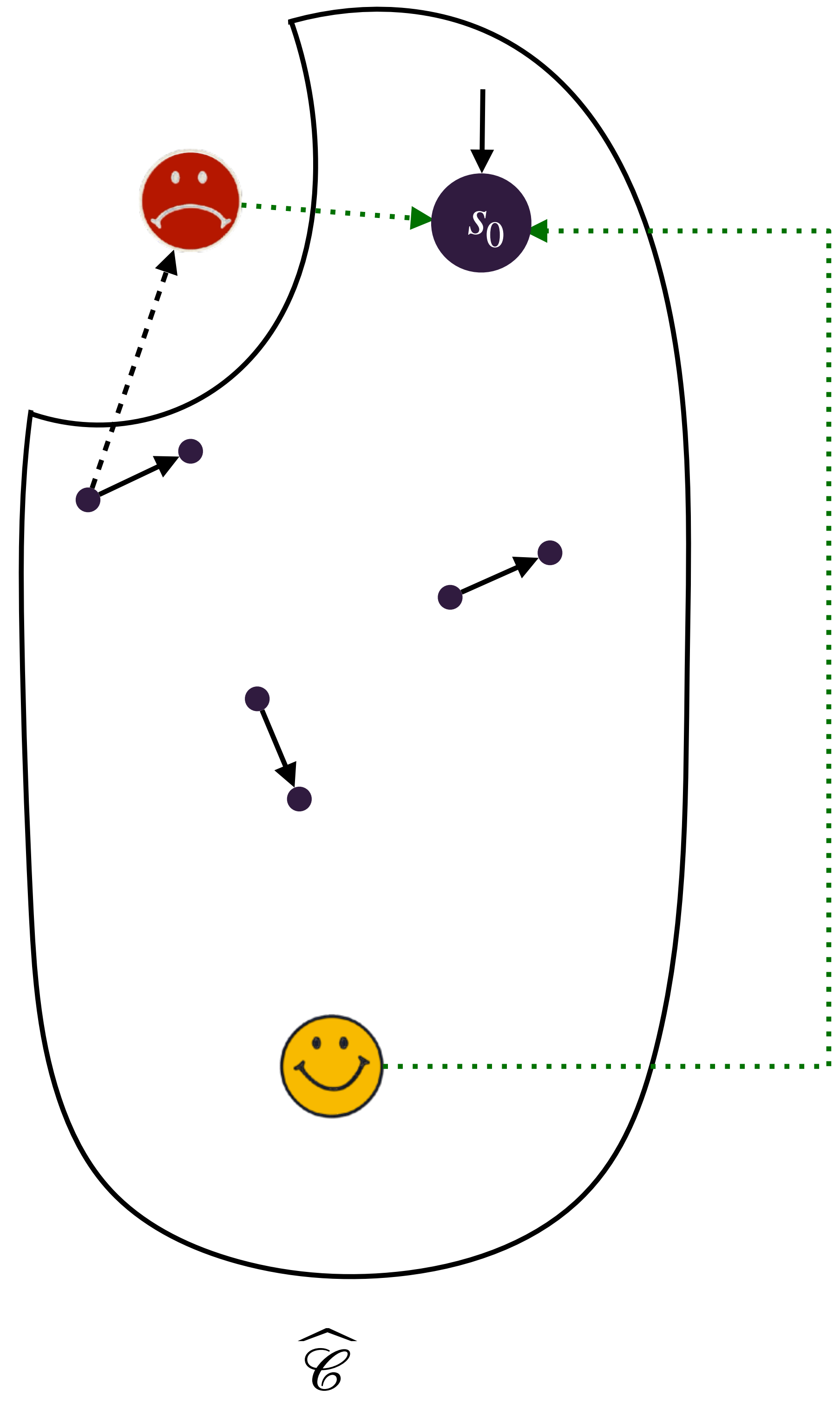

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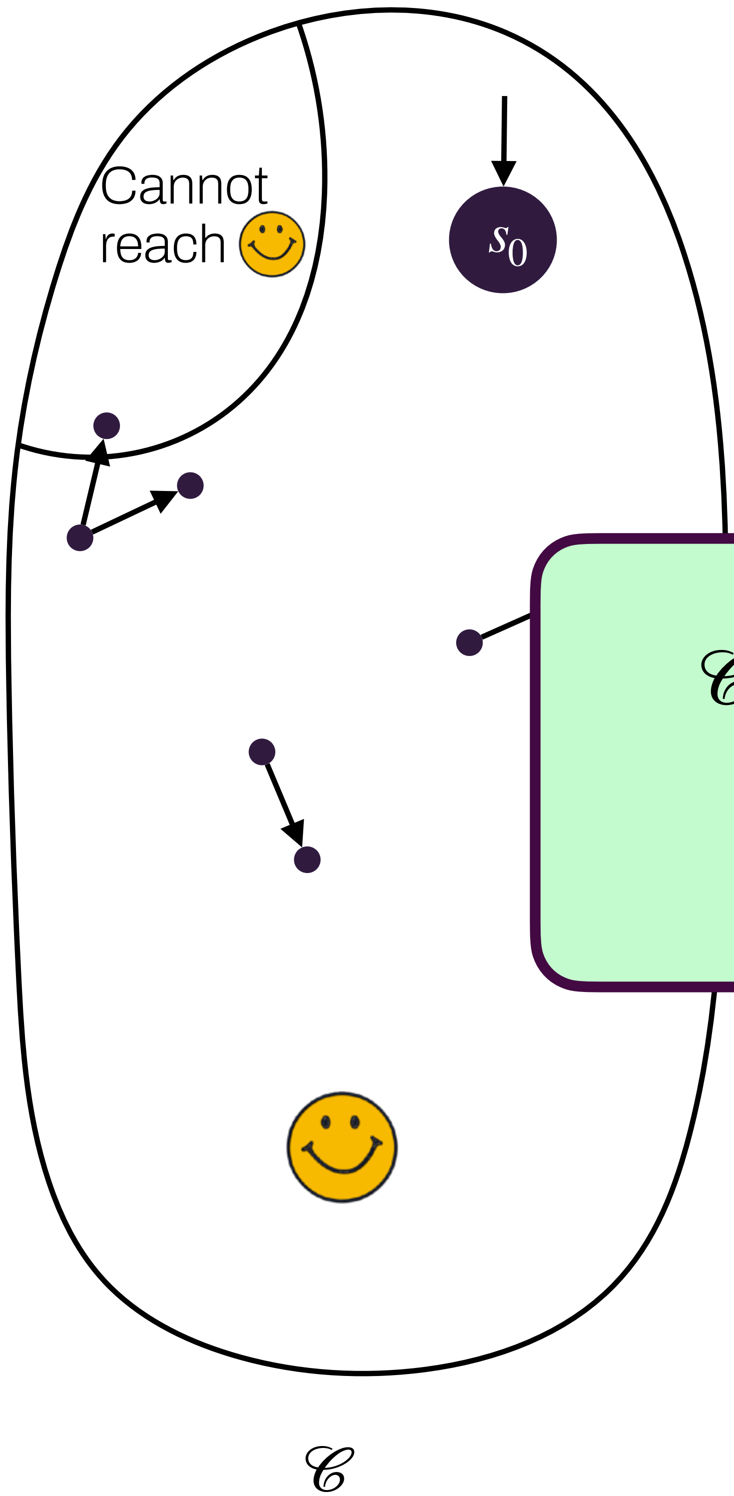


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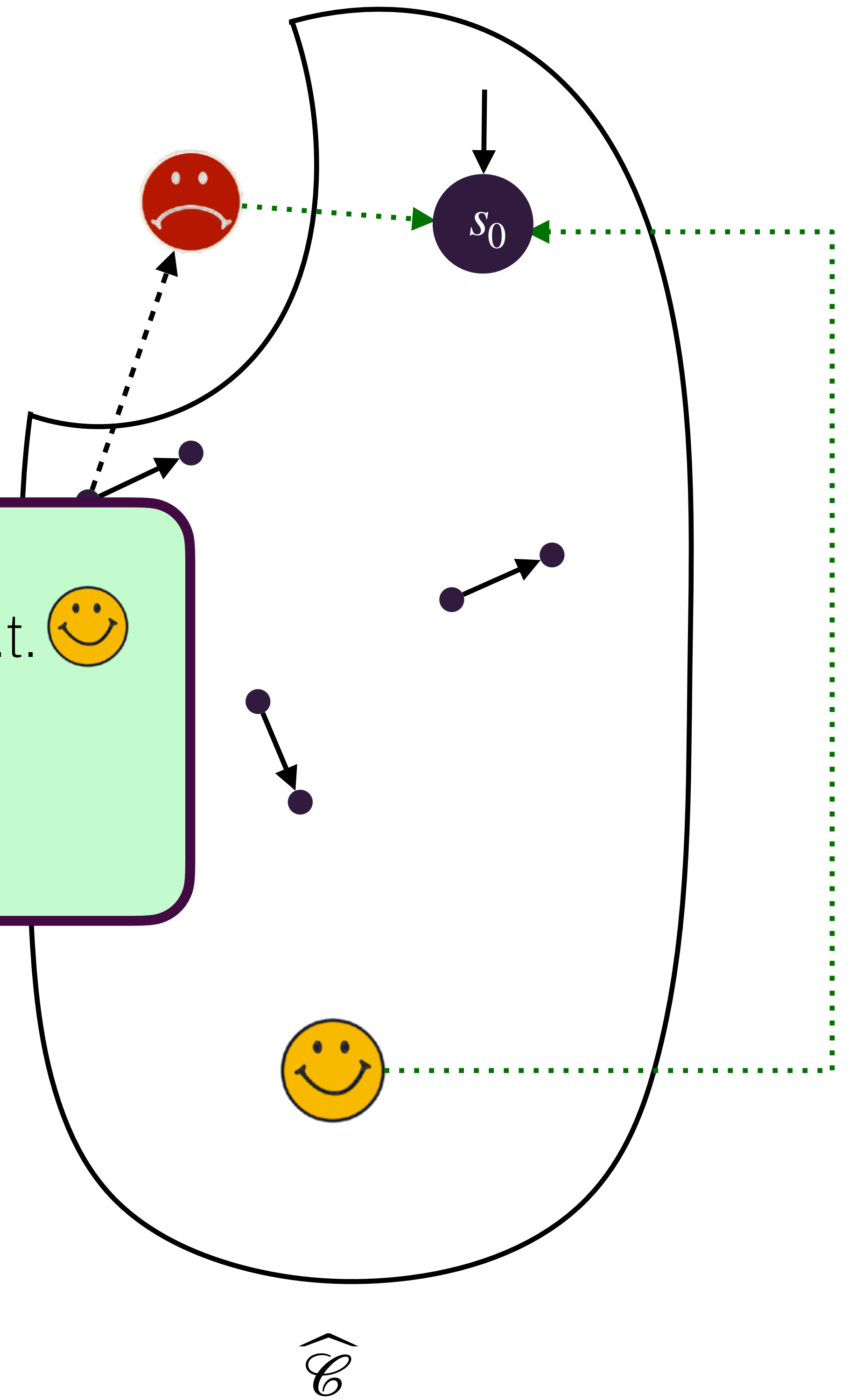

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The time to sample even increases/diverges!

# Statistical guarantees

**Hoeffding's inequalities**

Let $\varepsilon, \delta > 0$, let $N \geq \frac{8}{\varepsilon^2} \log\left(\frac{2}{\delta}\right)$. Then:

$$\mathbb{P}\left( \left| \frac{n_N}{N} - \mathbb{P}(\mathbf{F}\,\text{☺}) \right| \geq \frac{\varepsilon}{2} \right) \leq \delta$$

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Empirical average

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Empirical average

Fix two parameters, the third one follows

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# A slightly more general setting

- Given $L : S^+ \to \mathbb{R}$, the 😊-function $f_{L,😊}$ is $\mathbf{1}_{\mathbf{F}😊} \cdot L$
- We are interested in evaluating the quantity $\mathbb{E}(f_{L,😊})$
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Let $\varepsilon, \delta > 0$ s.t. $N \geq \dfrac{8B^2}{\varepsilon^2} \log\left(\dfrac{2}{\delta}\right)$. Then:

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Empirical estimation

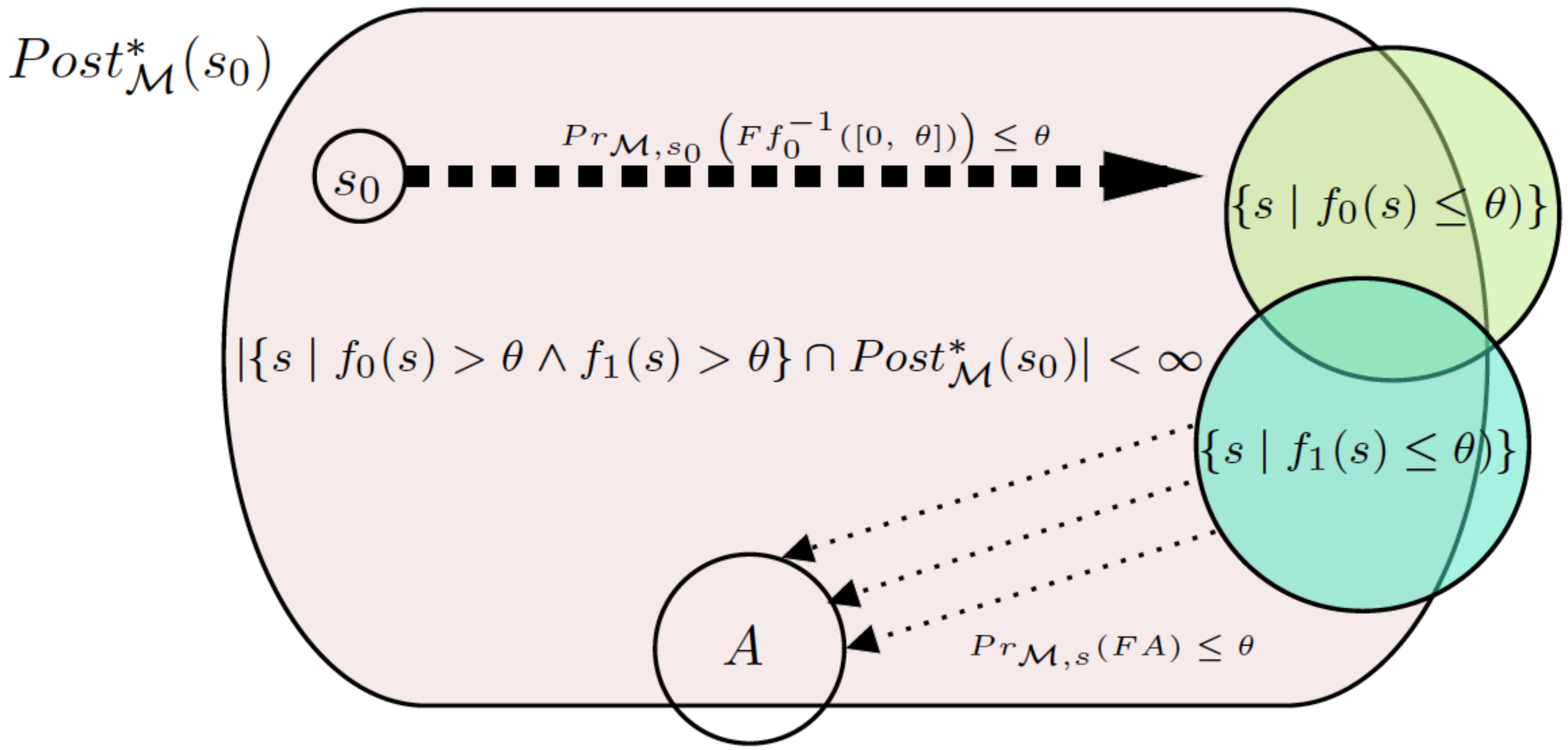
What can we do for
non-decisive Markov chains??

# Another numerical generic approach

## Divergent Markov Chains

A Markov chain $\mathcal{M}$ is *divergent* w.r.t. $s_0$ and $A$ if there exist two computable functions $f_0$ and $f_1$ from $S$ to $\mathbb{R}_{\geq 0}$ such that:

1. For all $0 < \theta < 1$, $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F} f_0^{-1}([0,\theta])) \leq \theta$;
2. For all $s \in S$, $\mathbf{Pr}_{\mathcal{M},s}(\mathbf{F} A) \leq f_1(s)$;
3. For all $0 < \theta < 1$, $\{s \mid f_0(s) > \theta \wedge f_1(s) > \theta\} \cap Post^*_{\mathcal{M}}(s_0)$ is finite.




[FHY23b] A. Finkel, S. Haddad, L. Ye. Introducing divergence for infinite probabilistic models (RP'23)

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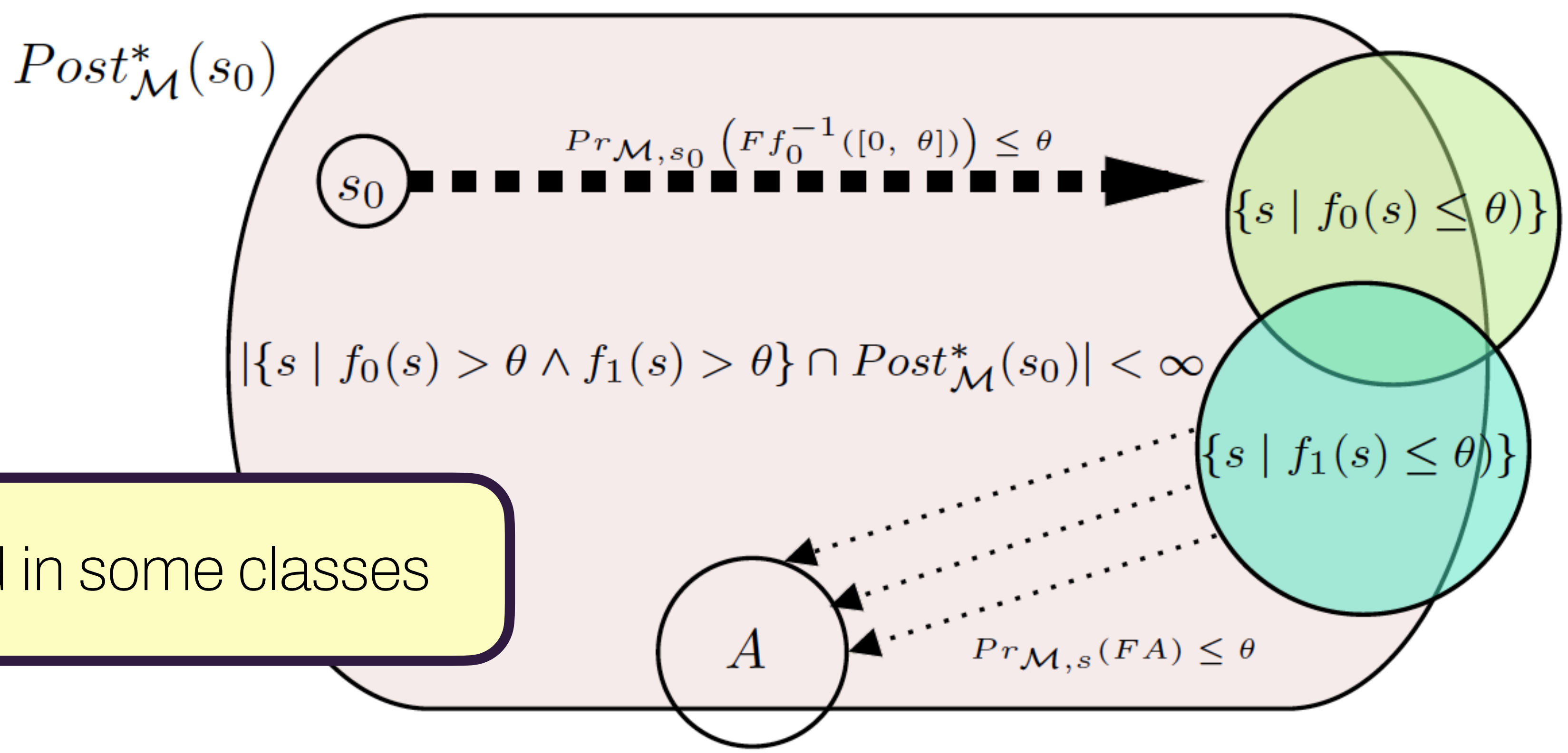


Can be decided in some classes

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# Importance sampling for rare events evaluation

- Issue: rare events in $\mathscr{C}$

## Rare-Event Problem for Statistical Model Checking

### Problem Statement

- We want to estimate the probability of a rare event $e$ occurring with probability close to $10^{-15}$.
- We want a *confidence level* of 0.99.
- We are able to compute $10^9$ trajectories.

### Possible Outcomes

| Number of occurrences of $e$ | Probability | Confidence interval |
|---|---|---|
| 0 | $\approx 1 - 10^{-6}$ | $[0, 7.03 \cdot 10^{-9}]$ |
| 1 | $\leq 10^{-6}$ | $[6.83 \cdot 10^{-10}, 1.69 \cdot 10^{-9}]$ |
| $n > 1$ | $\leq 10^{-12}$ | $> 6.83 \cdot 10^{-10}$ |

[Bar14] B. Barbot. Acceleration for statistical model checking (PhD thesis)
[BHP12] B. Barbot, S. Haddad, C. Picaronny. Coupling and Importance Sampling for Statistical Model Checking (TACAS'12)

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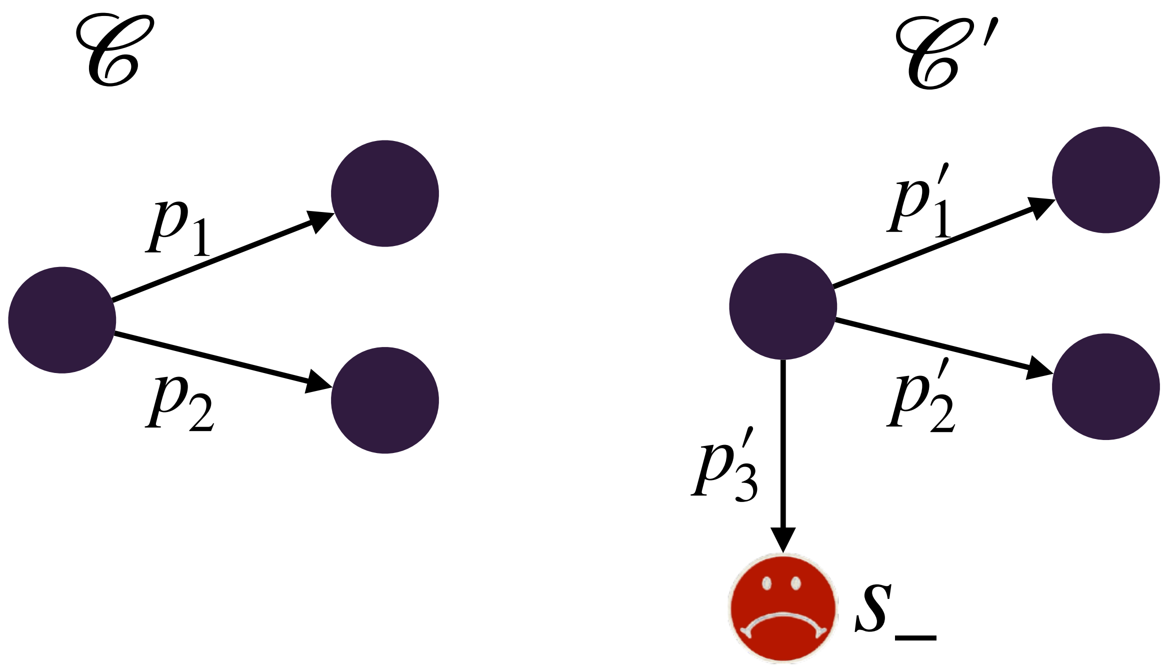


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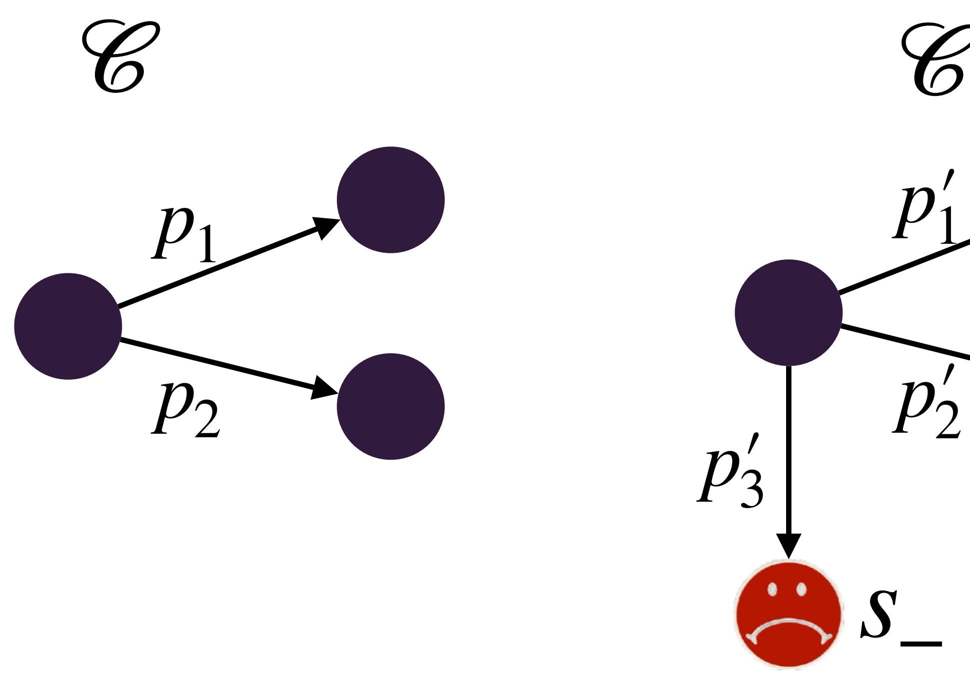
**Likelihood and biased function**

$$\gamma(\rho) = \begin{cases} \dfrac{P(\rho)}{P'(\rho)} & \text{if } \rho \text{ ends in } ☺ \\ 0 & \text{otherwise} \end{cases}$$

$$L' = L \cdot \gamma$$





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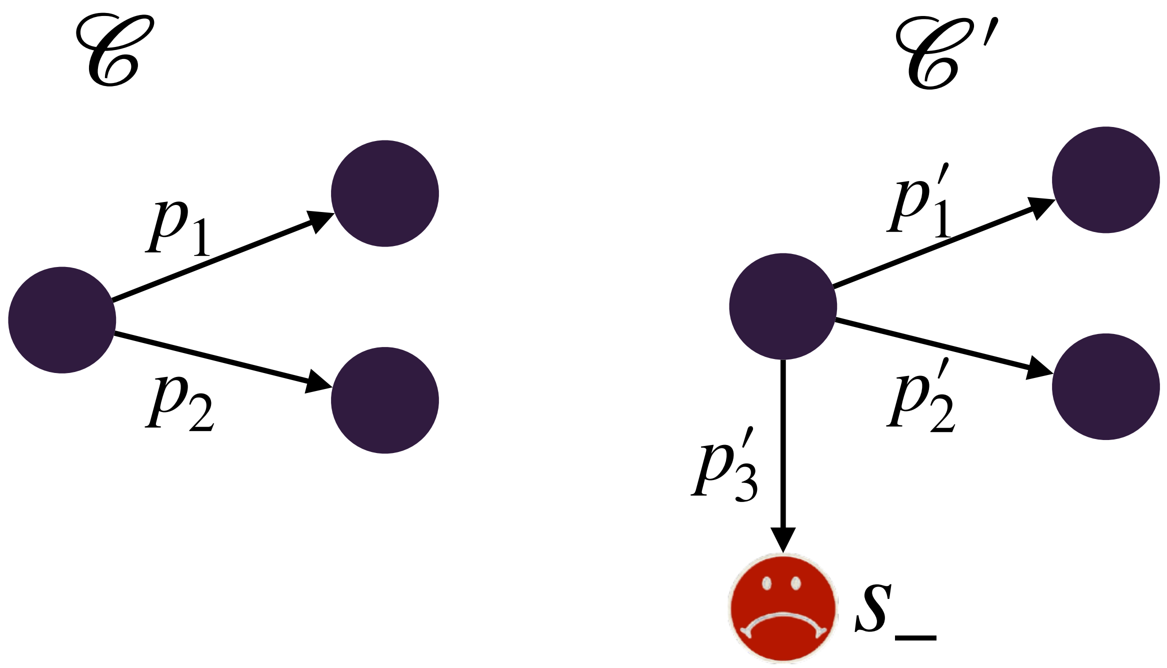



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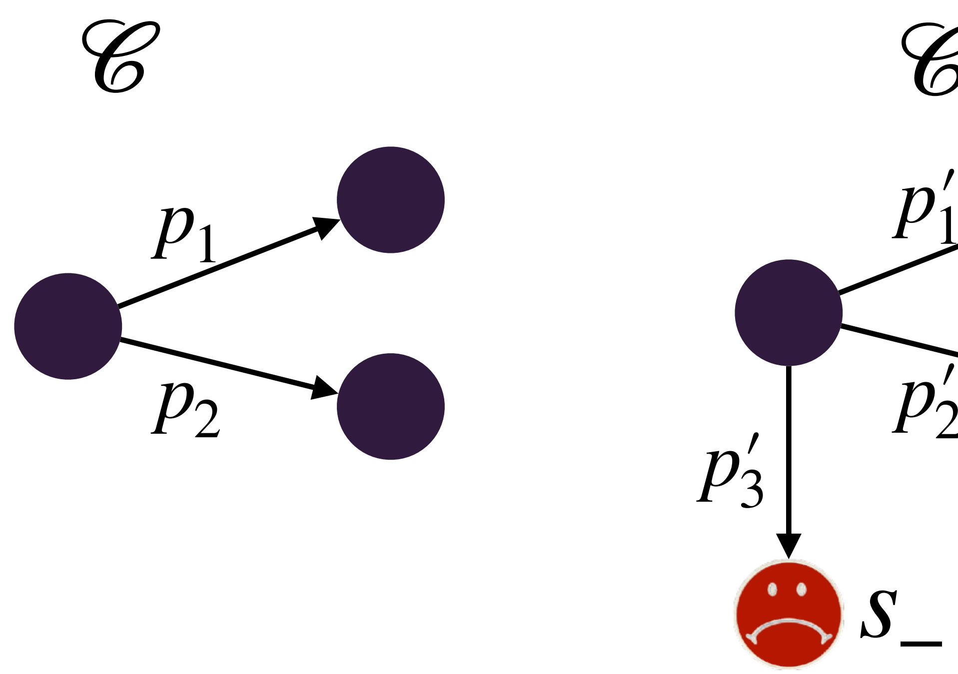


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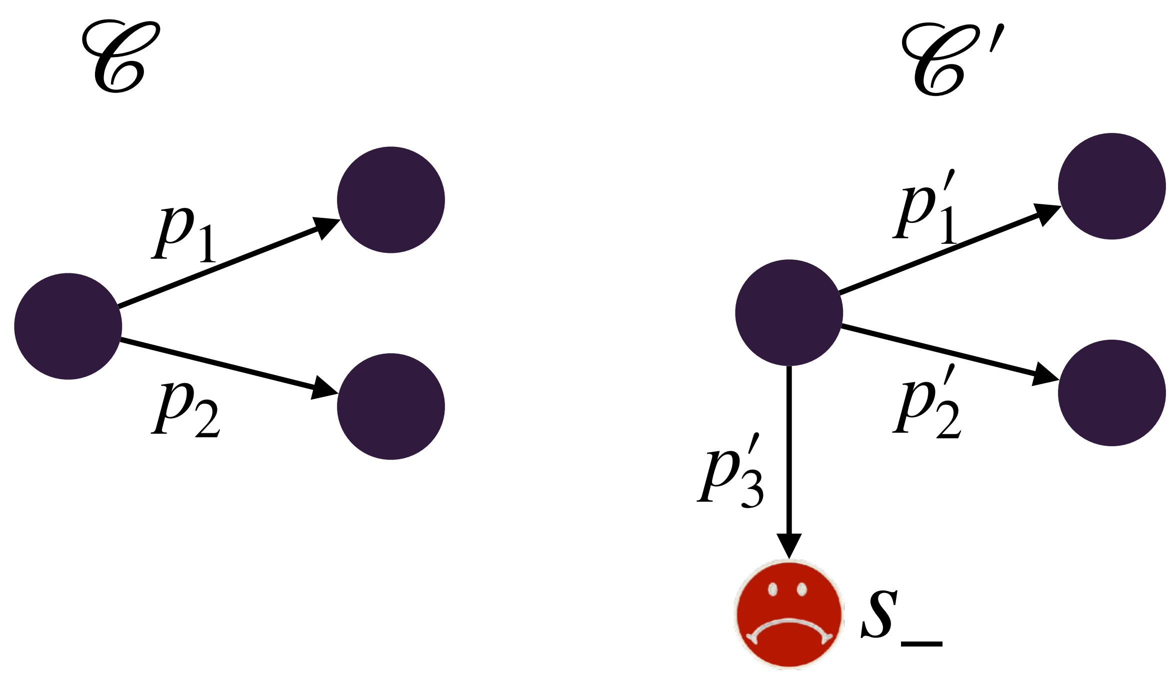


+ setting giving statistical guarantees

[Bar14] B. Barbot. Acceleration for statistical model checking (PhD thesis)
[BHP12] B. Barbot, S. Haddad, C. Picaronny. Coupling and Importance Sampling for Statistical Model Checking (TACAS'12)

# Importance sampling for rare events evaluation

- Issue: rare events in $\mathscr{C}$
- Idea: analyze a biased Markov chain $\mathscr{C}'$, which amplifies the « rare » event

**Likelihood and biased function**

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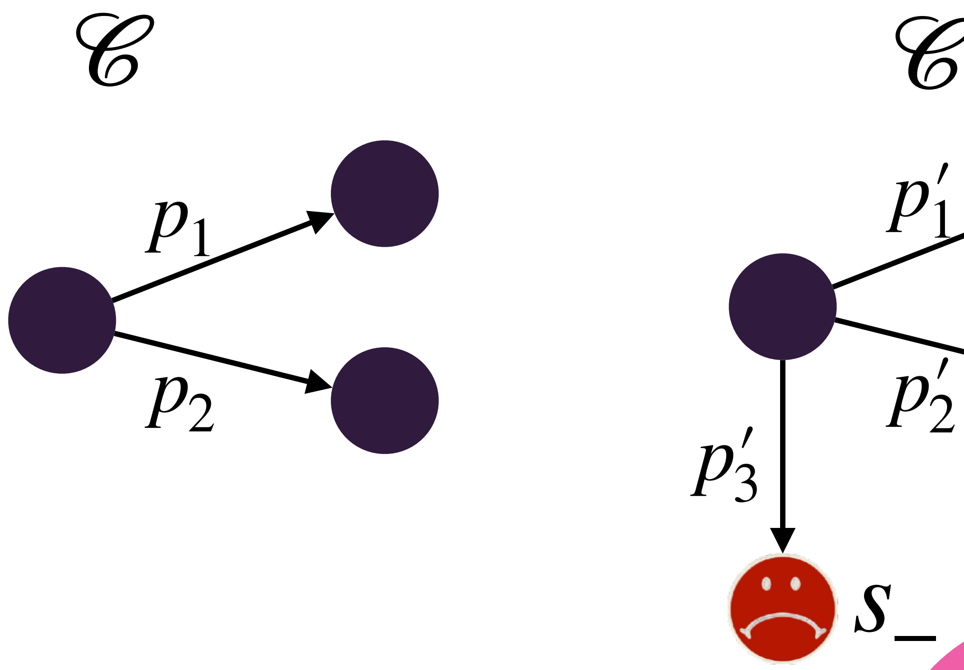


Before that, only estimators!!

+ setting giving statistical guarantees

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We propose to use the importance sampling approach to analyze some non-decisive DTMCs!

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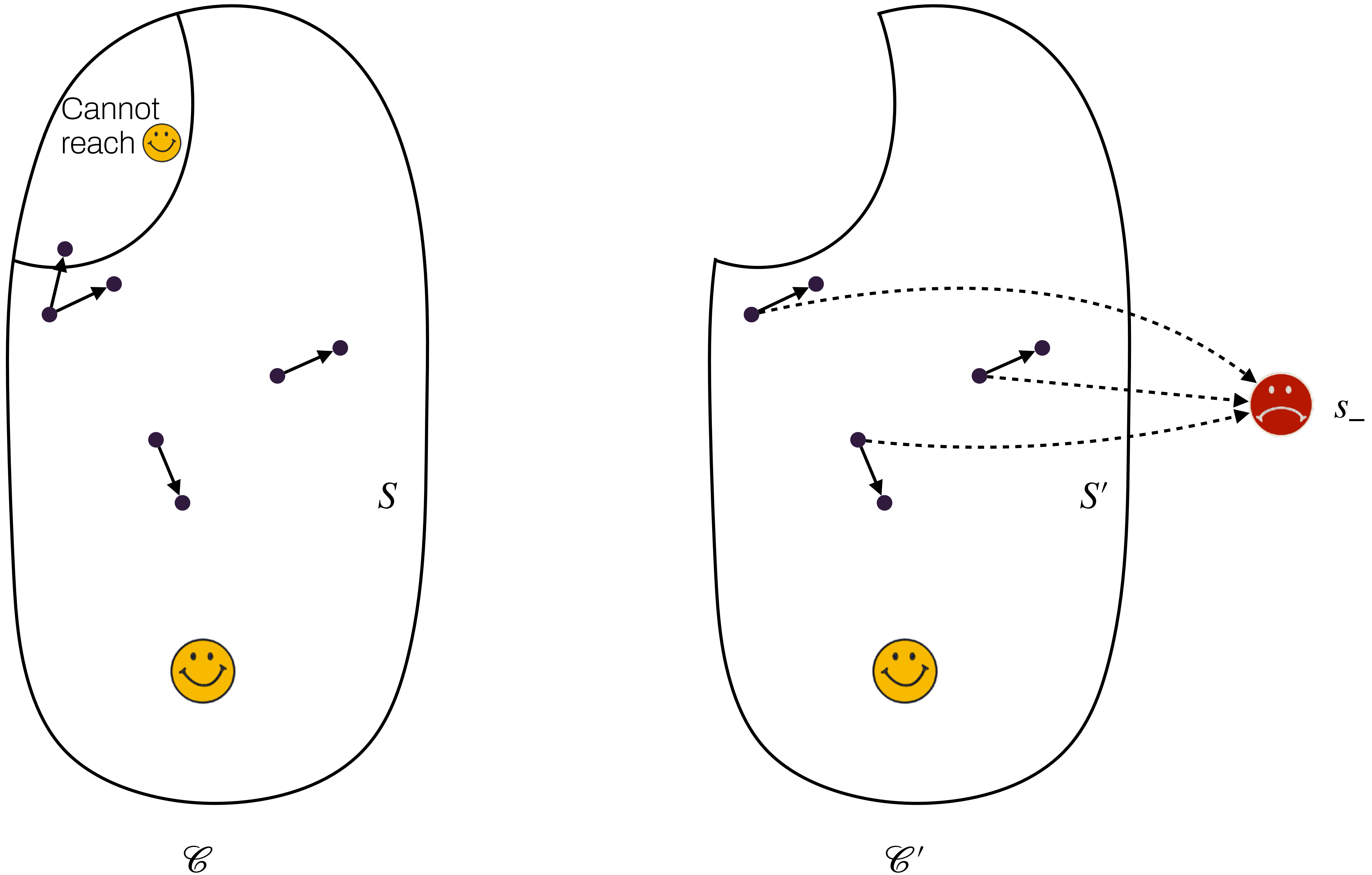
First time that importance sampling is used not to accelerate the analysis, but to enable the analysis

# Biased Markov chain



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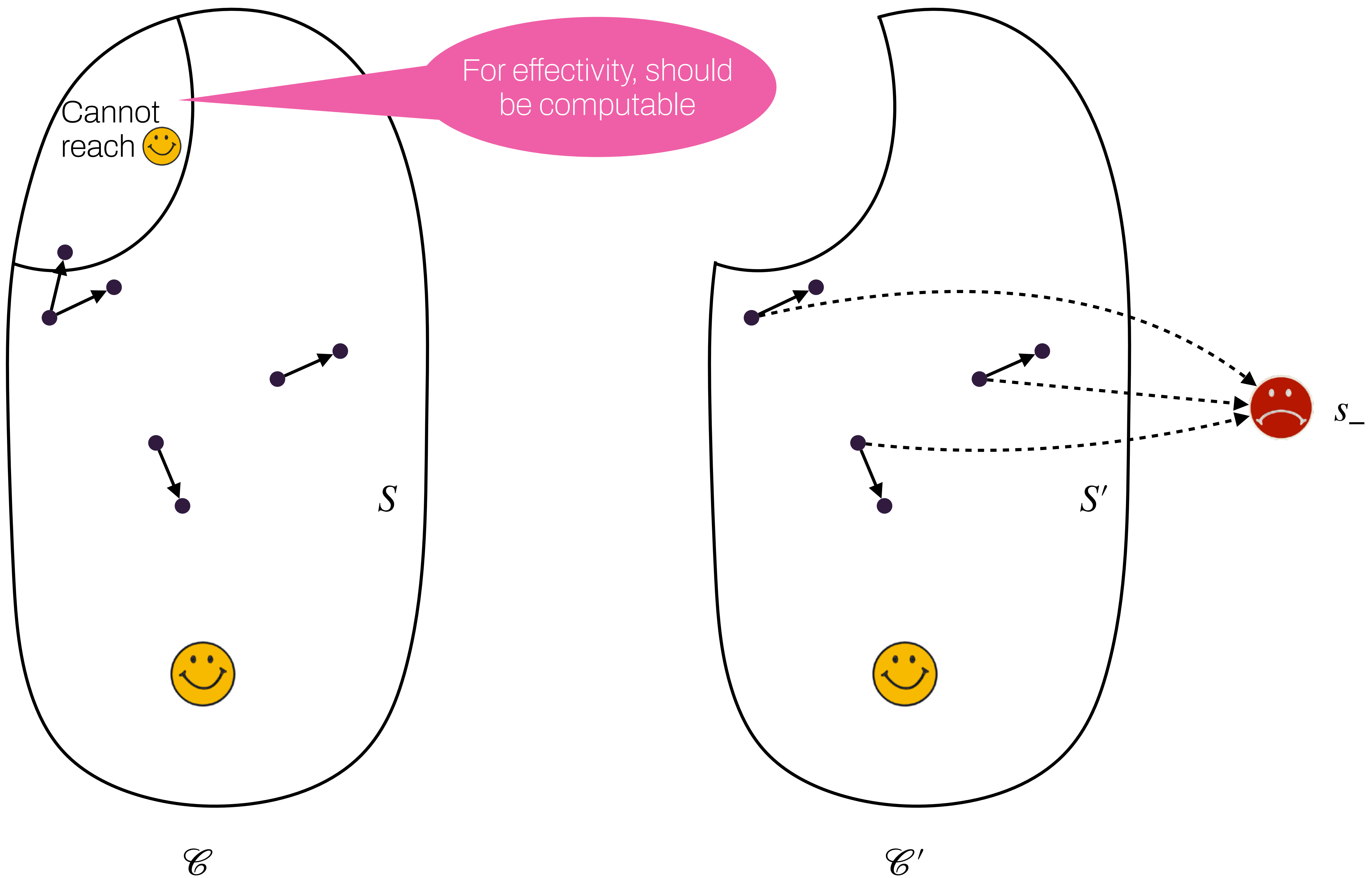

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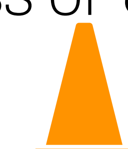
  - $L'$ can be unbounded even if $L$ is bounded 
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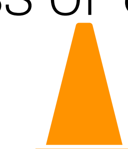
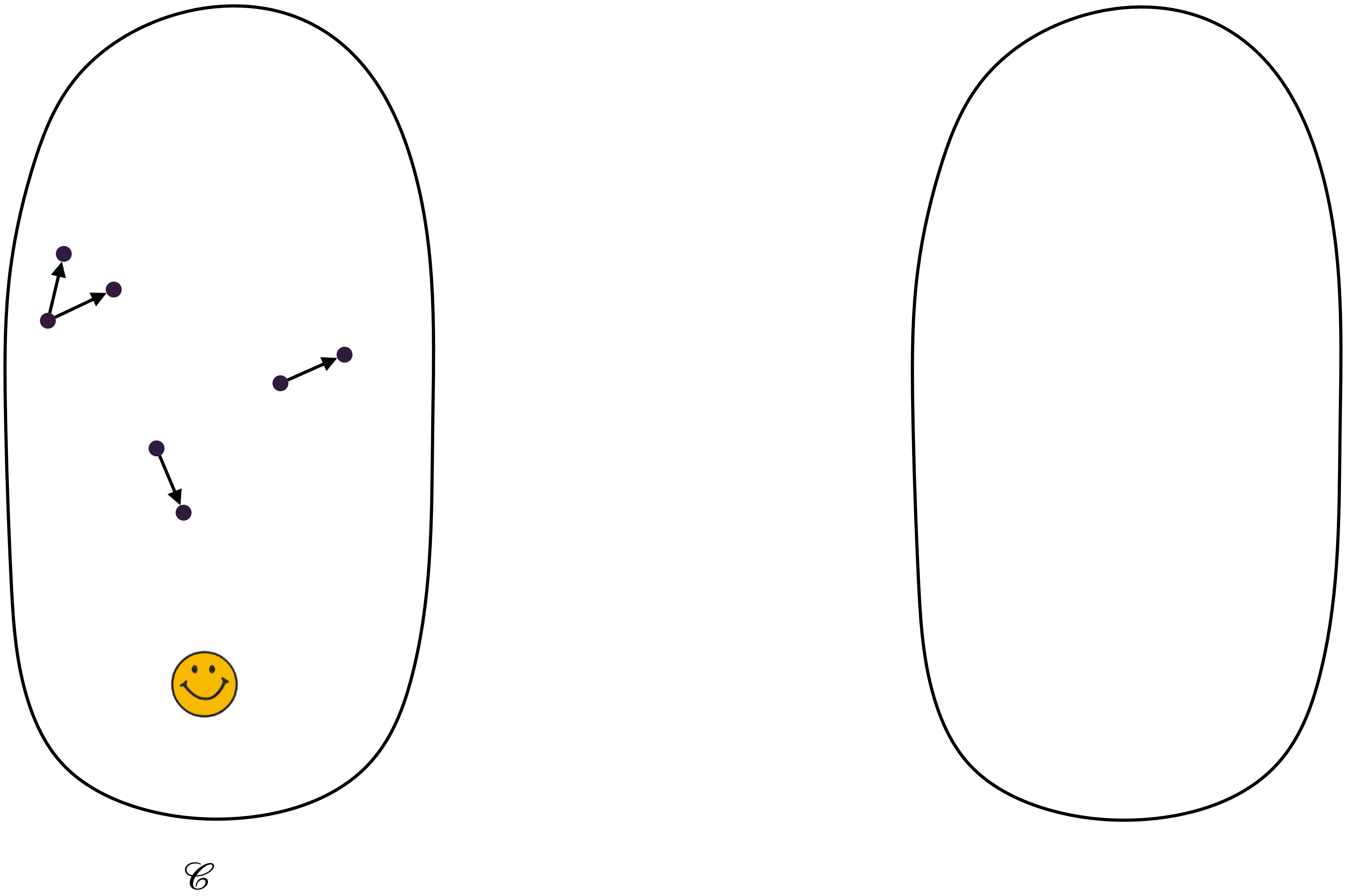
  - $L'$ can be unbounded even if $L$ is bounded 
- Need of developing methods to ensure nice properties of $\mathscr{C}'$
  - [BHP12] for rare events: approach for finite Markov chains via coupling and abstractions with reduced variance

[BHP12] B. Barbot, S. Haddad, C. Picaronny. *Coupling and Importance Sampling for Statistical Model Checking (TACAS'12)*

# Our approach



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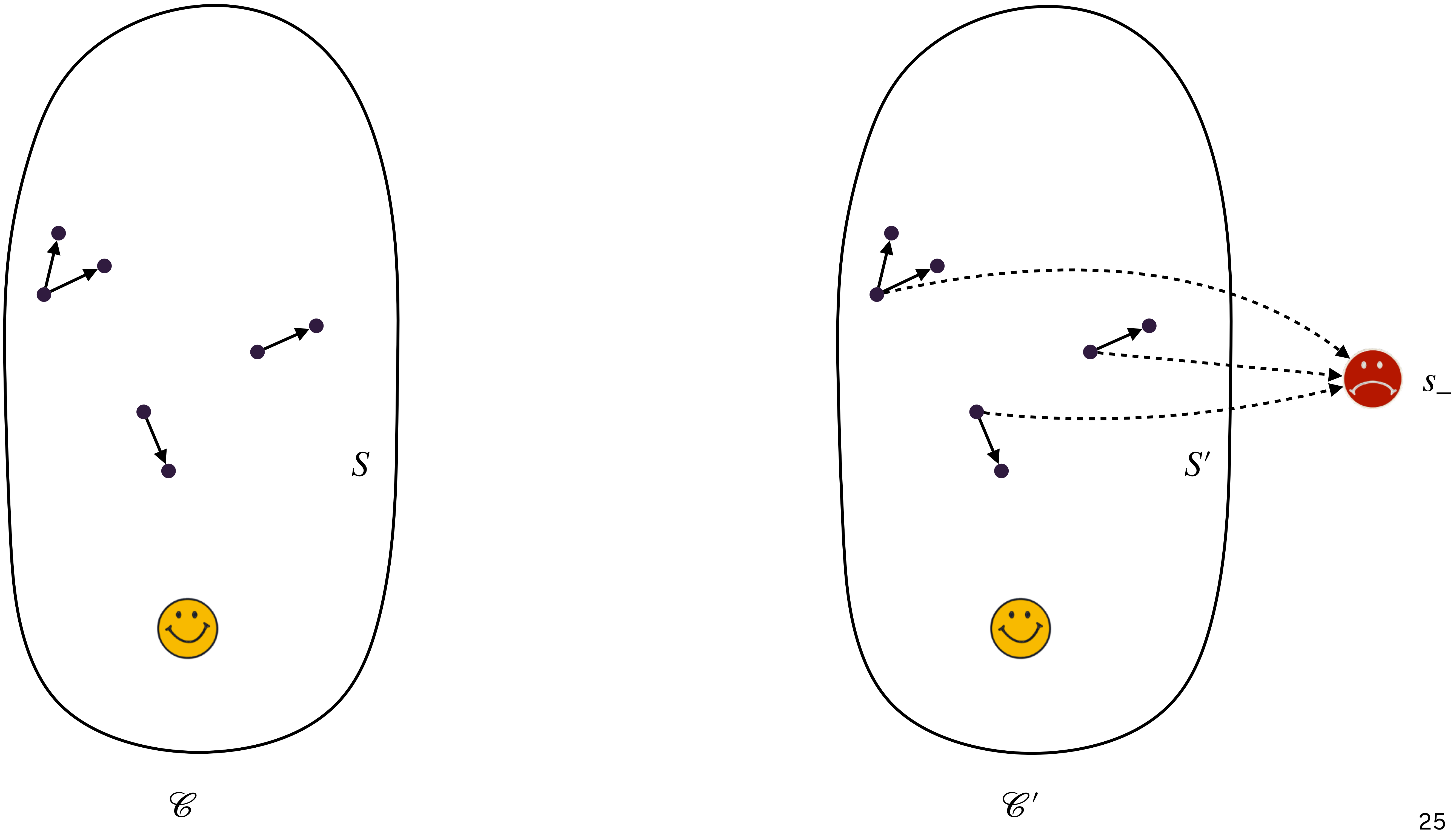

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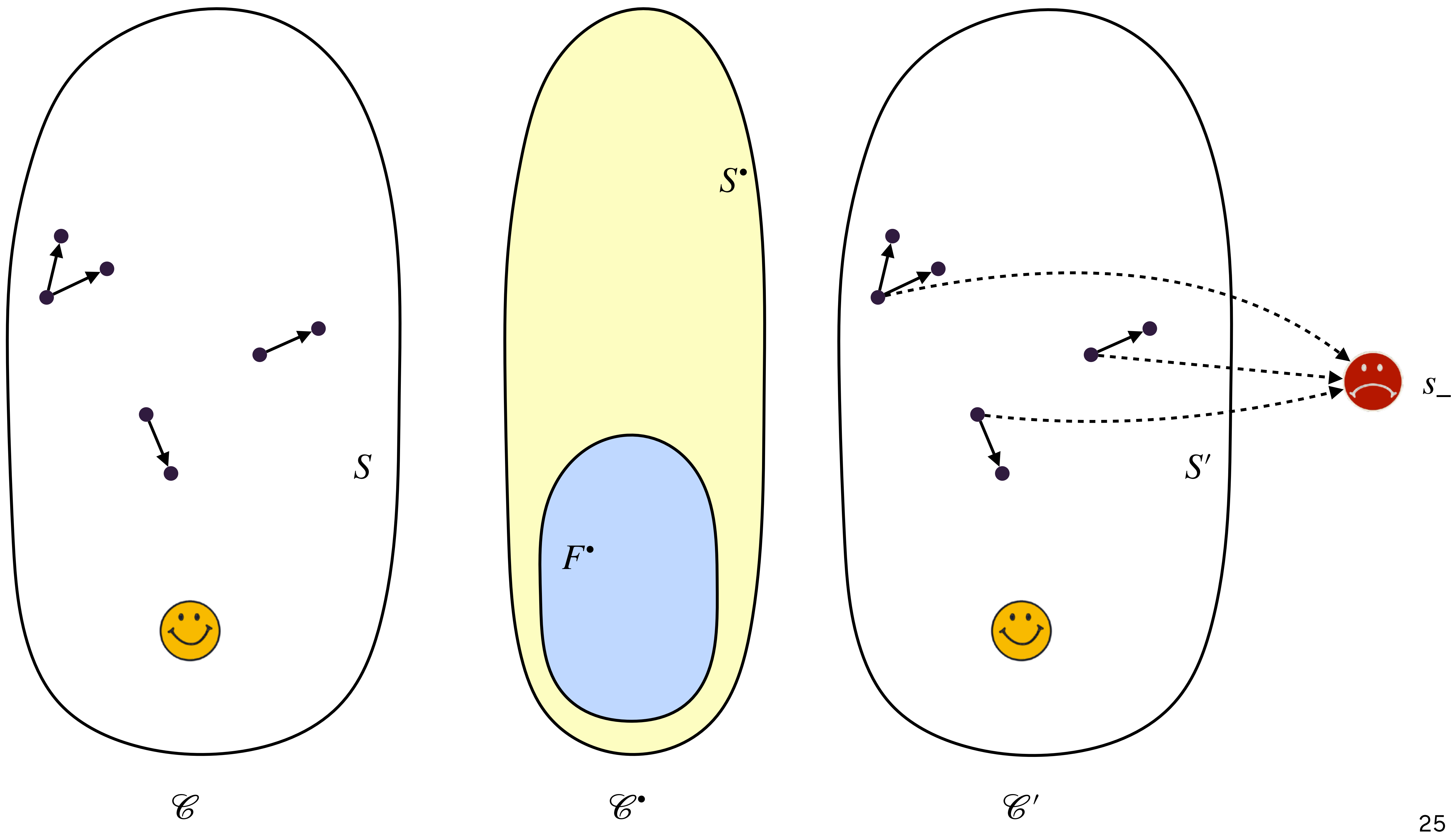

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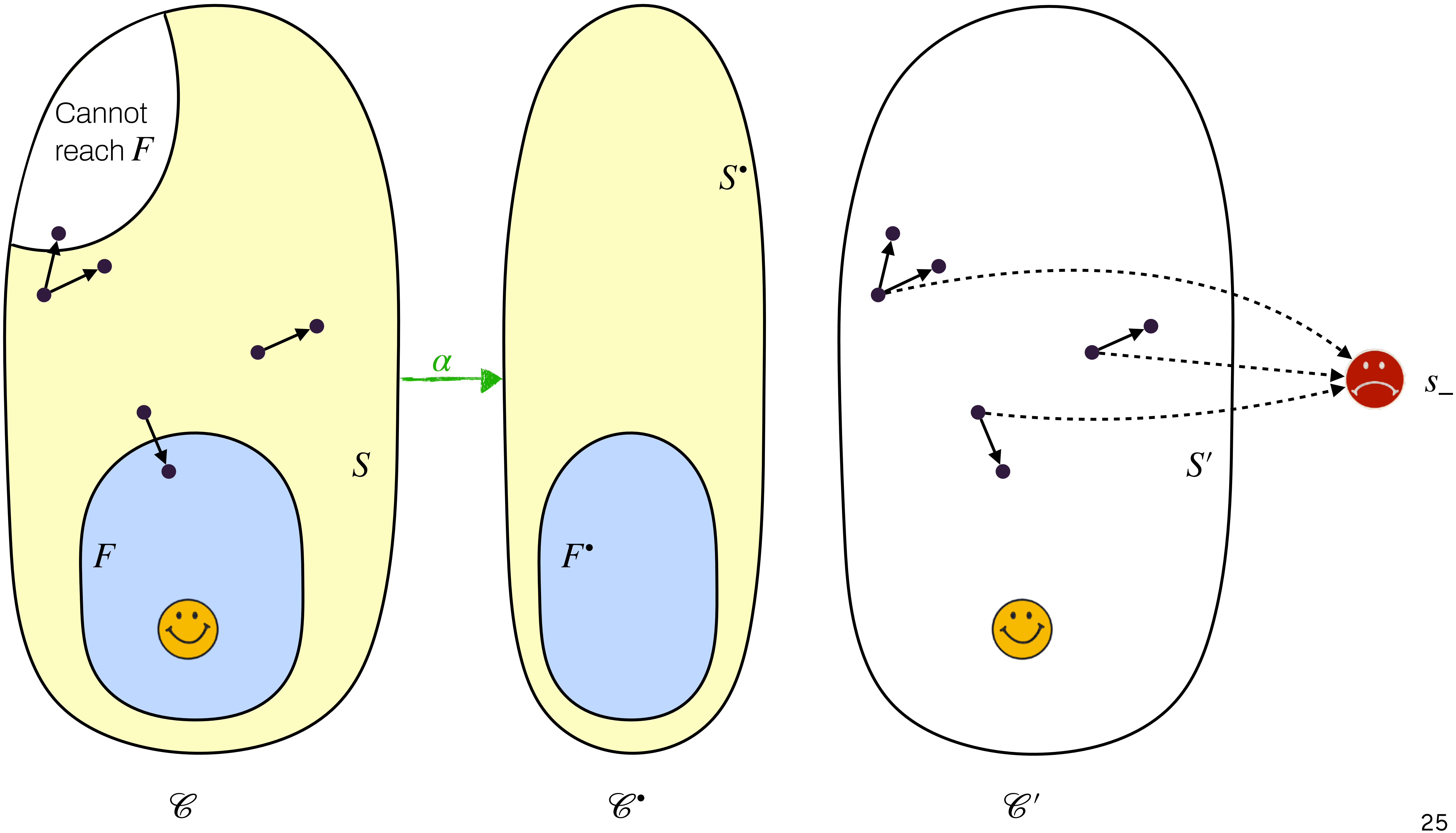

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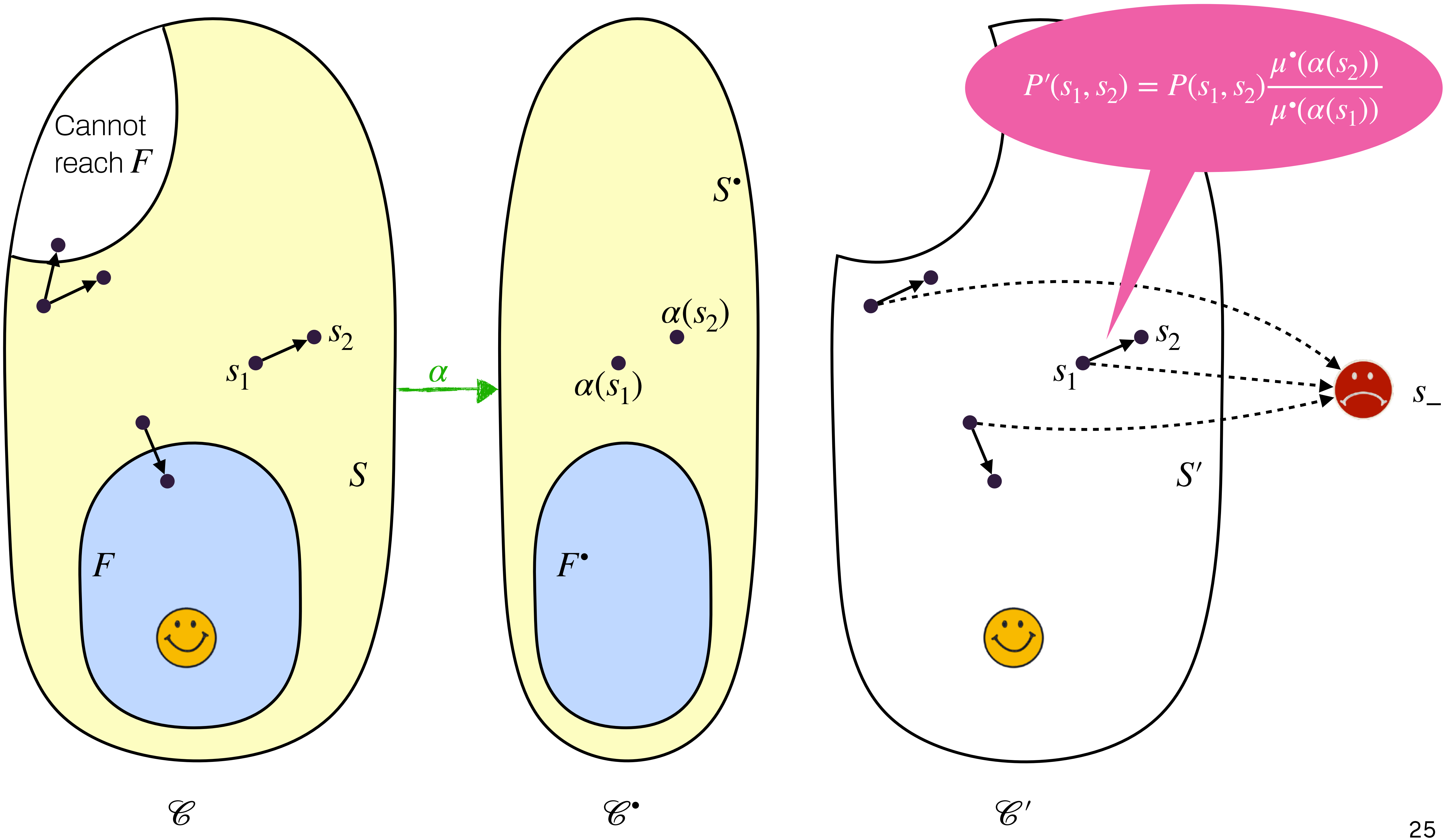

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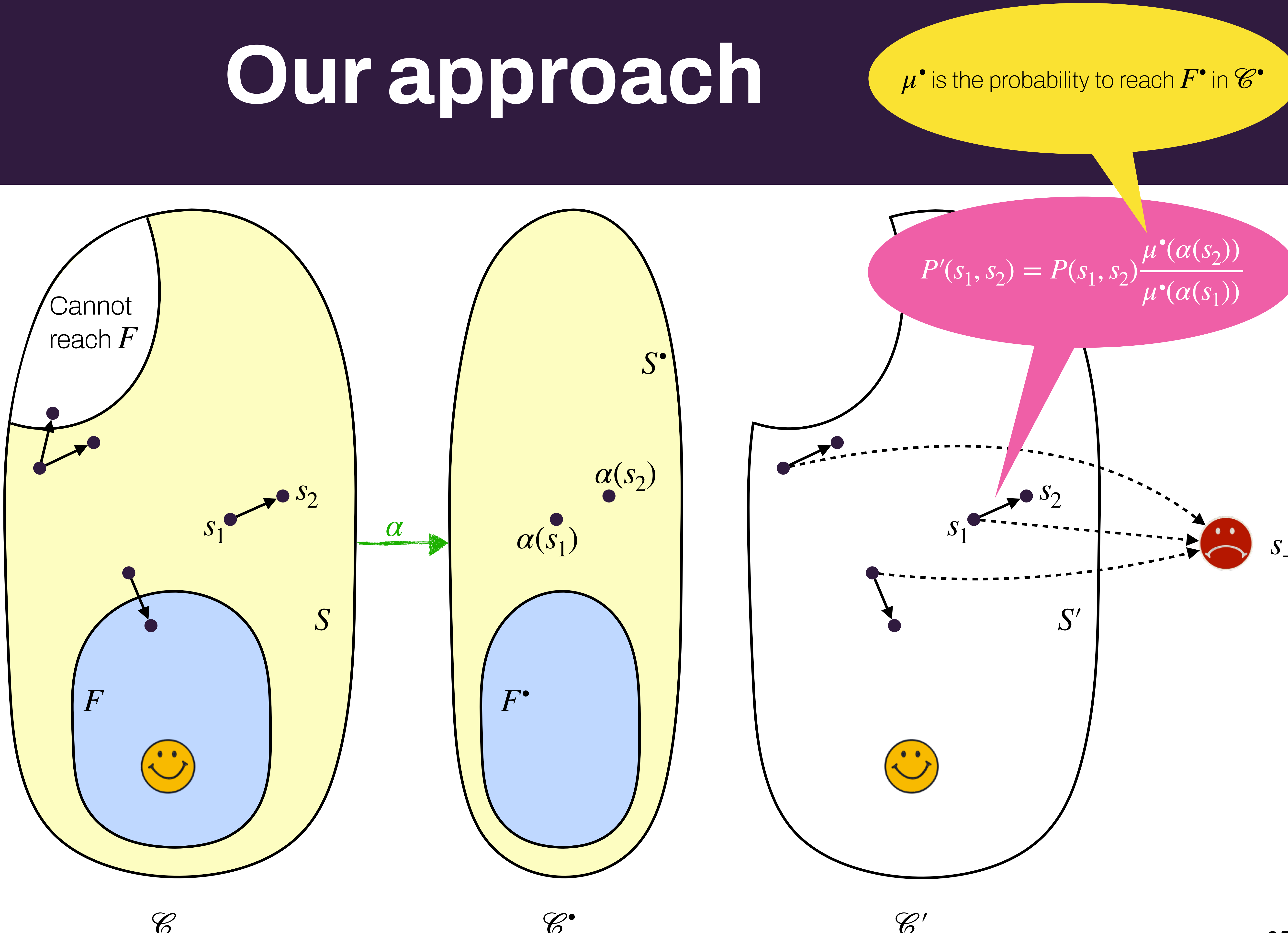

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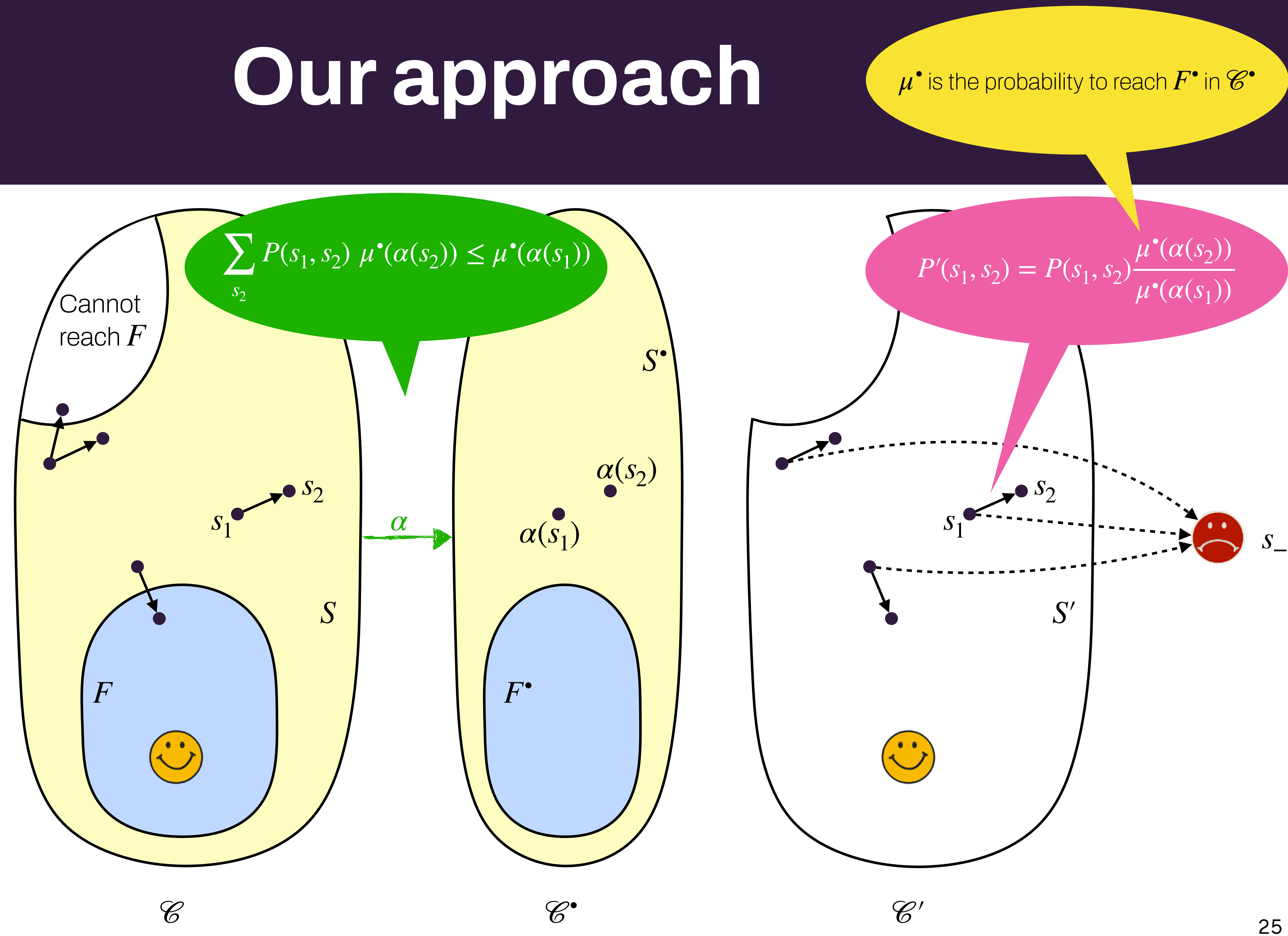

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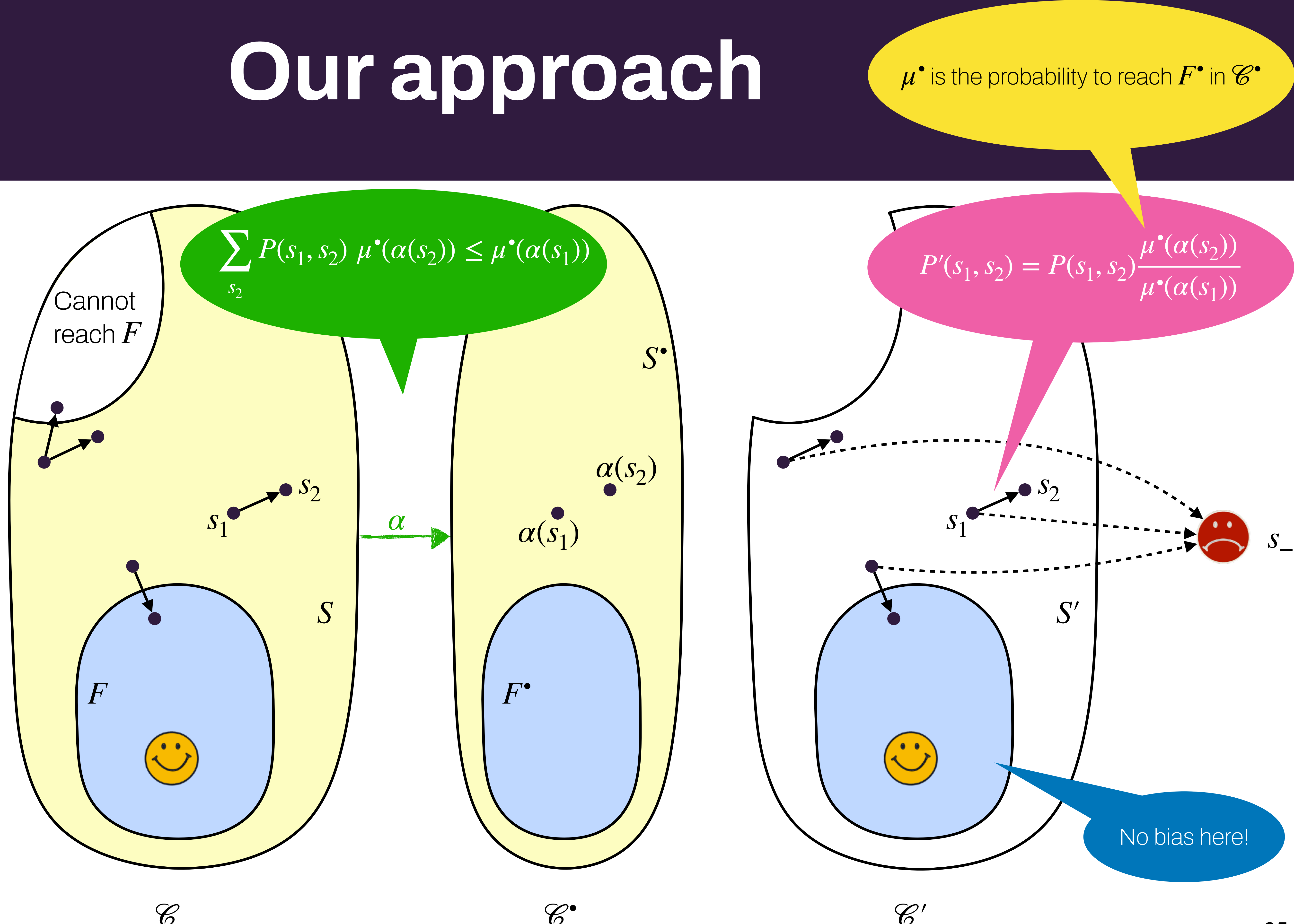

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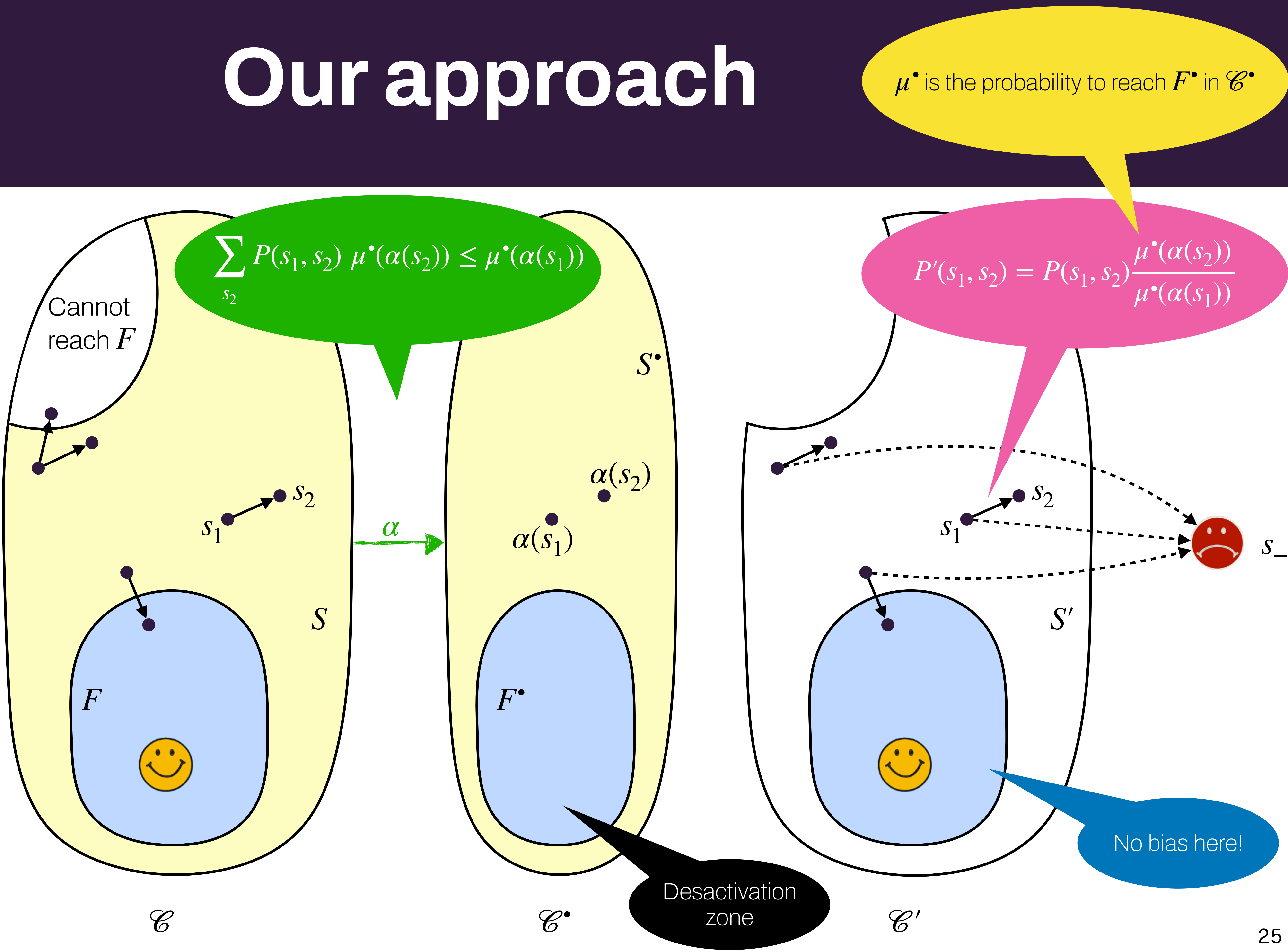

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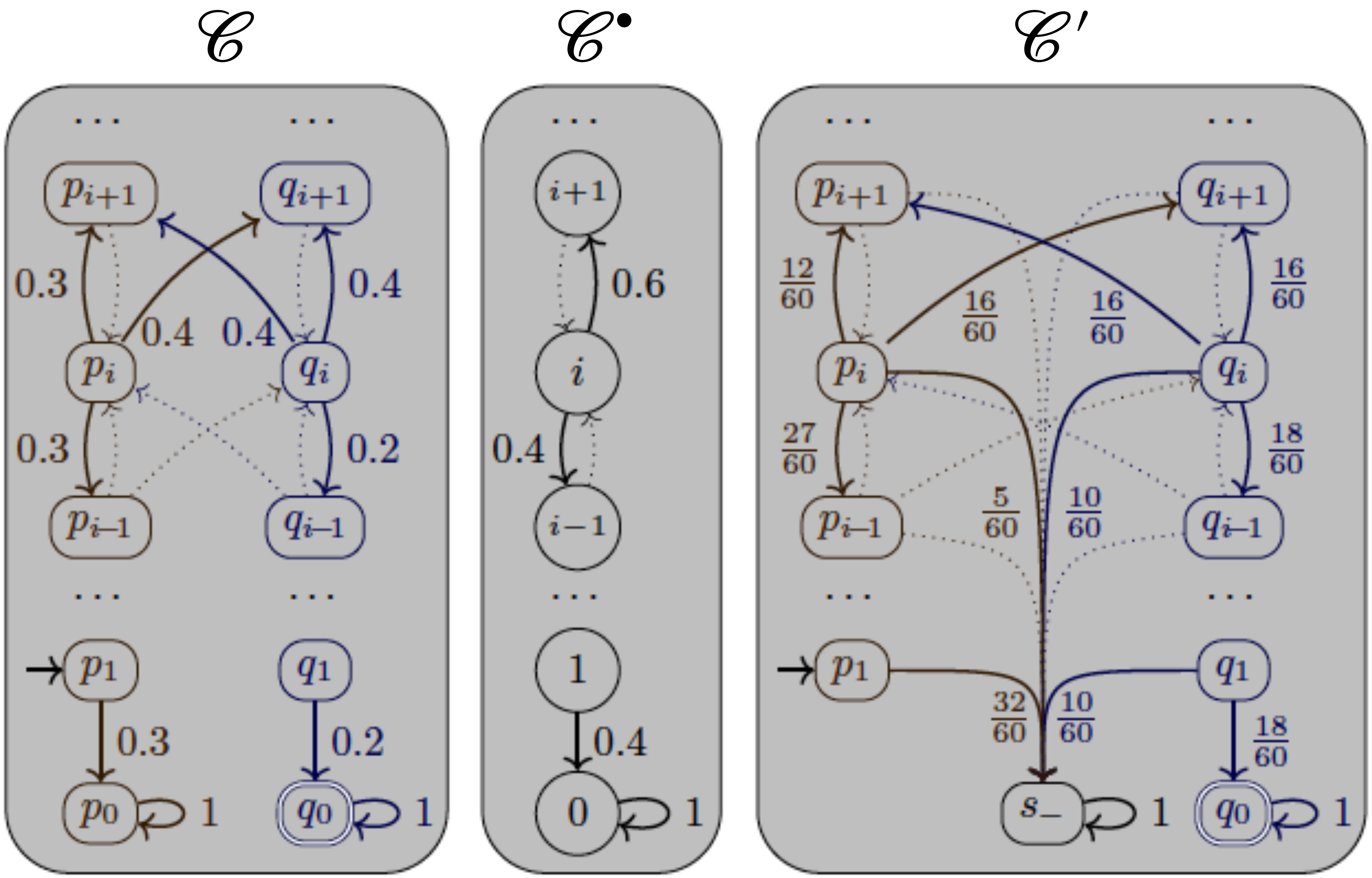
- We need:
  - To ensure the decisiveness of $\mathscr{C}'$
  - To compute $\mu^{\bullet}(\,\cdot\,)$ (useful in two places: to sample paths and to compute the final value when hitting ☺)

# Role of $F$

- Standard approach for importance sampling: no set $F$
  ($F$ coincides with  )
- Will be useful to adjust the properties satisfied by the abstraction to be correct
  - Requirement will be « outside $F$ »
  - For instance, congestion of systems

# Example




$F = \{☺\} = \{q_0\}$

$\mathscr{C}'$ is decisive

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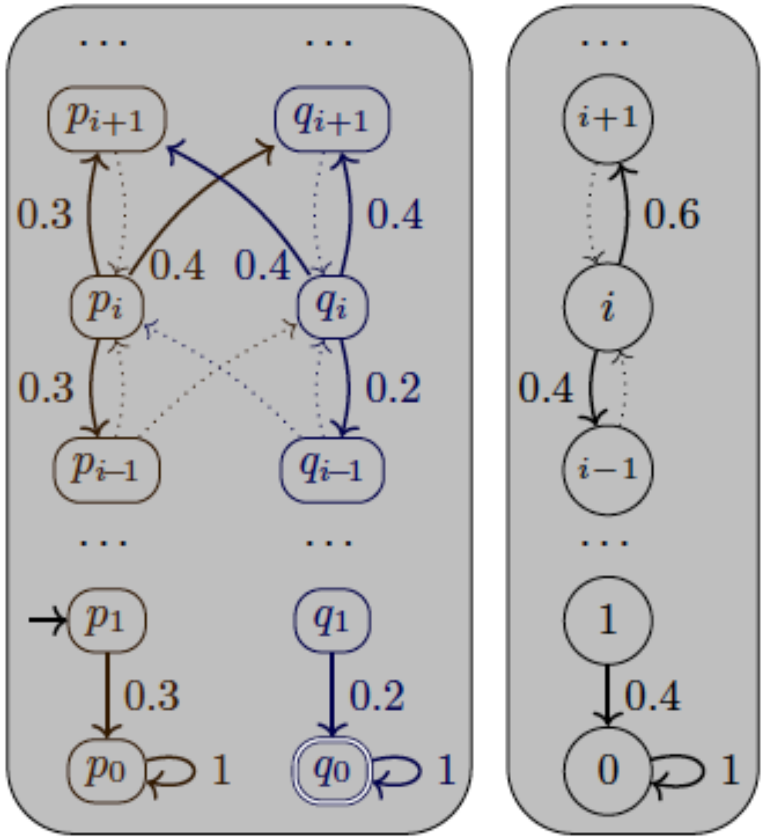

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+ generalization (written by Serge yesterday)

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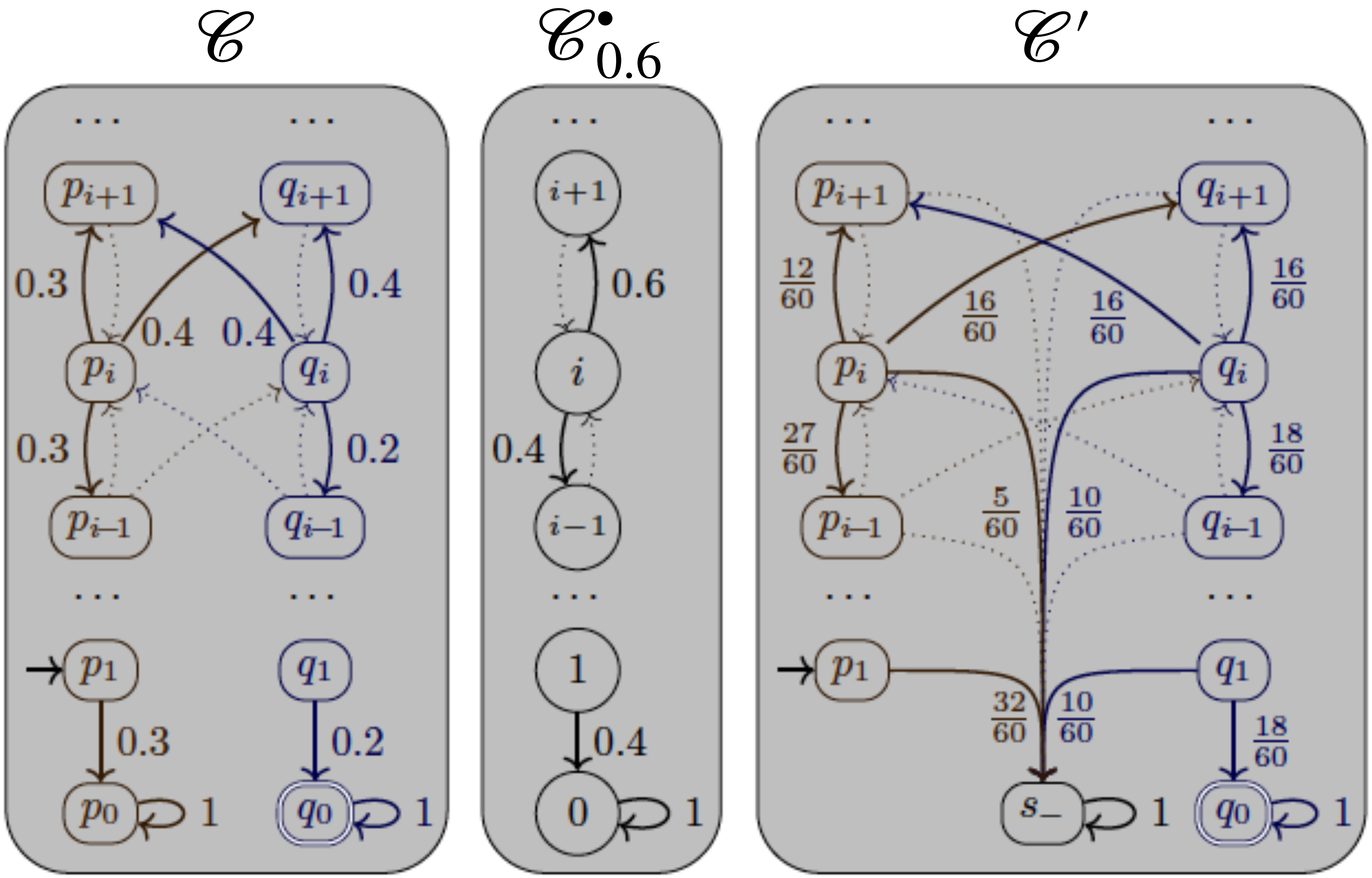


𝒞 is not decisive

𝒞′ is decisive

# Application to probabilistic pushdown automata

[EKM06] J. Esparza, A. Kucera, R. Mayr. Model checking pushdown automatas (LMCS)

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- Local conditions on transition rules to ensure the hypotheses of the theorem

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# Implementation

*https://cosmos.lacl.fr/*
*[BBDHP15] P. Ballarini, B. Barbot, M. Duflot, S. Haddad, N. Pekergin. Hasl: A new approach for performance evaluation and model checking from concepts to experimentation (Performance Evaluation)*

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- $p$ big: large desactivation zone ($N_0$)
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Note: in all experiments, the confidence is set to $99\ \%$

# First example

- State-free proba. pushdown automaton $\mathscr{C}$
  $A \xrightarrow{1} C \quad A \xrightarrow{n} BB \quad B \xrightarrow{5} \varepsilon$
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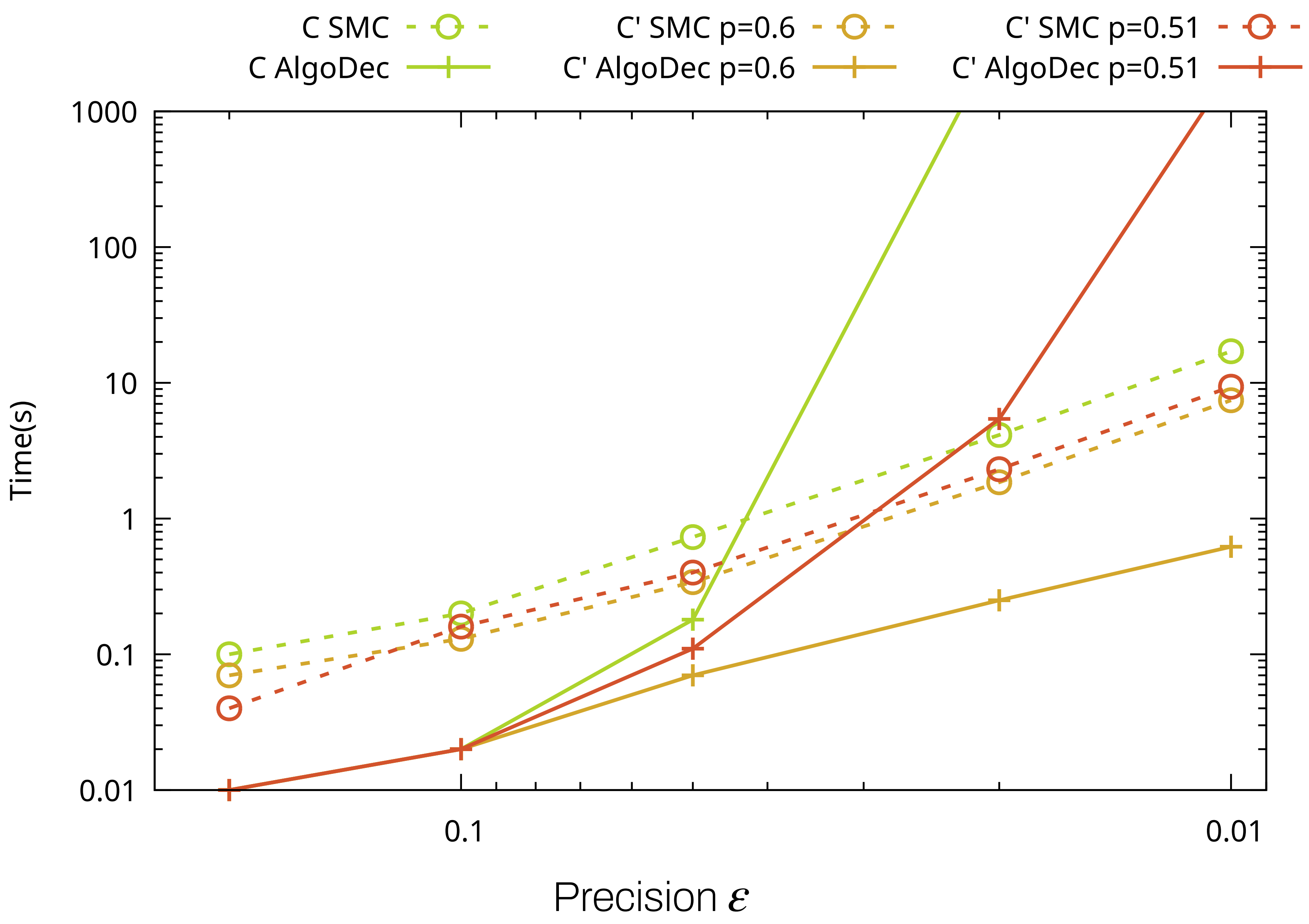
- It is decisive
- It is $p$-divergent for every $1/2 < p < 1$

# First example
# Experimental results



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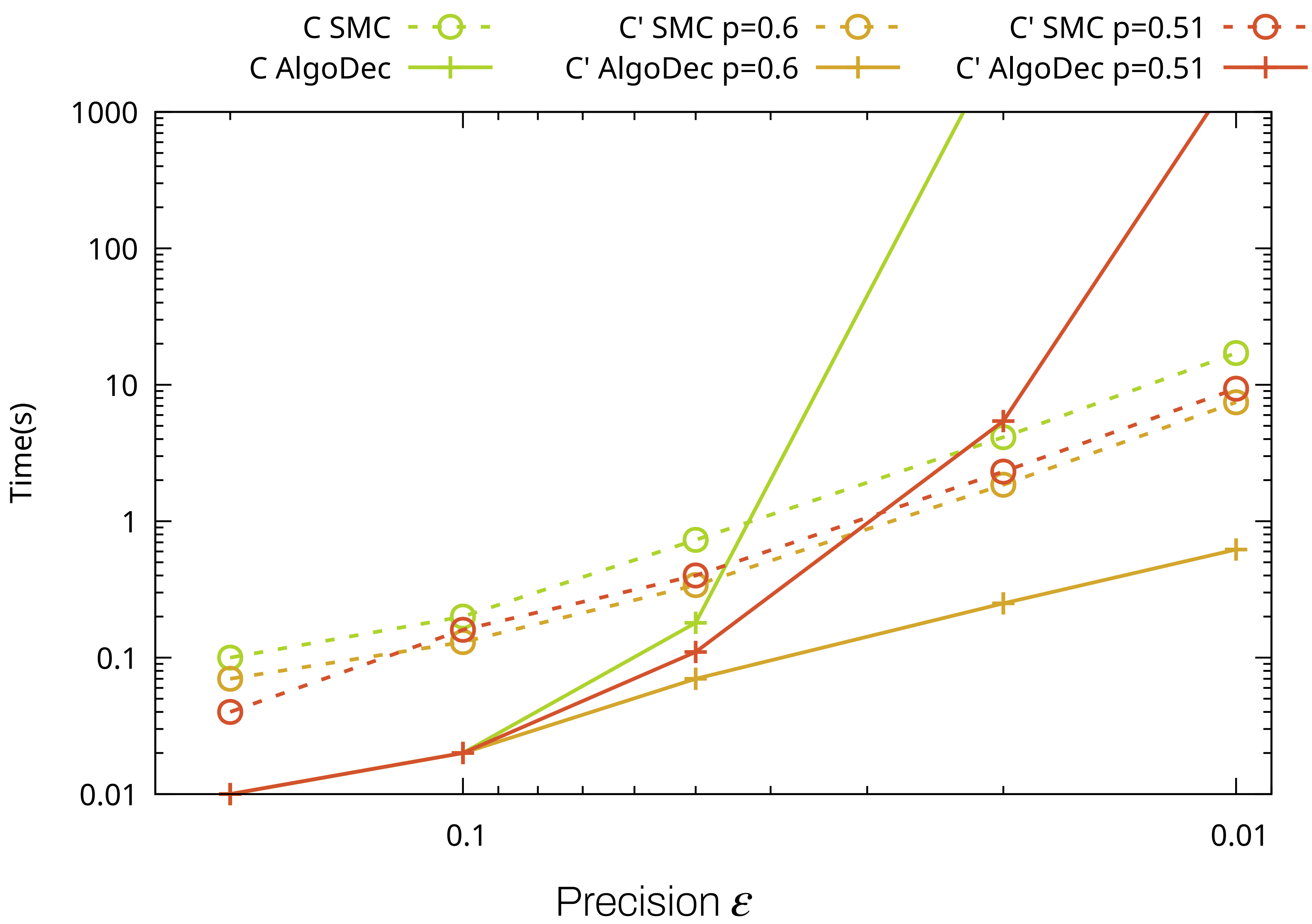


- In Estim (SMC): doubling the precision impacts in square on computation time (slope $2$ in this log-log scale)

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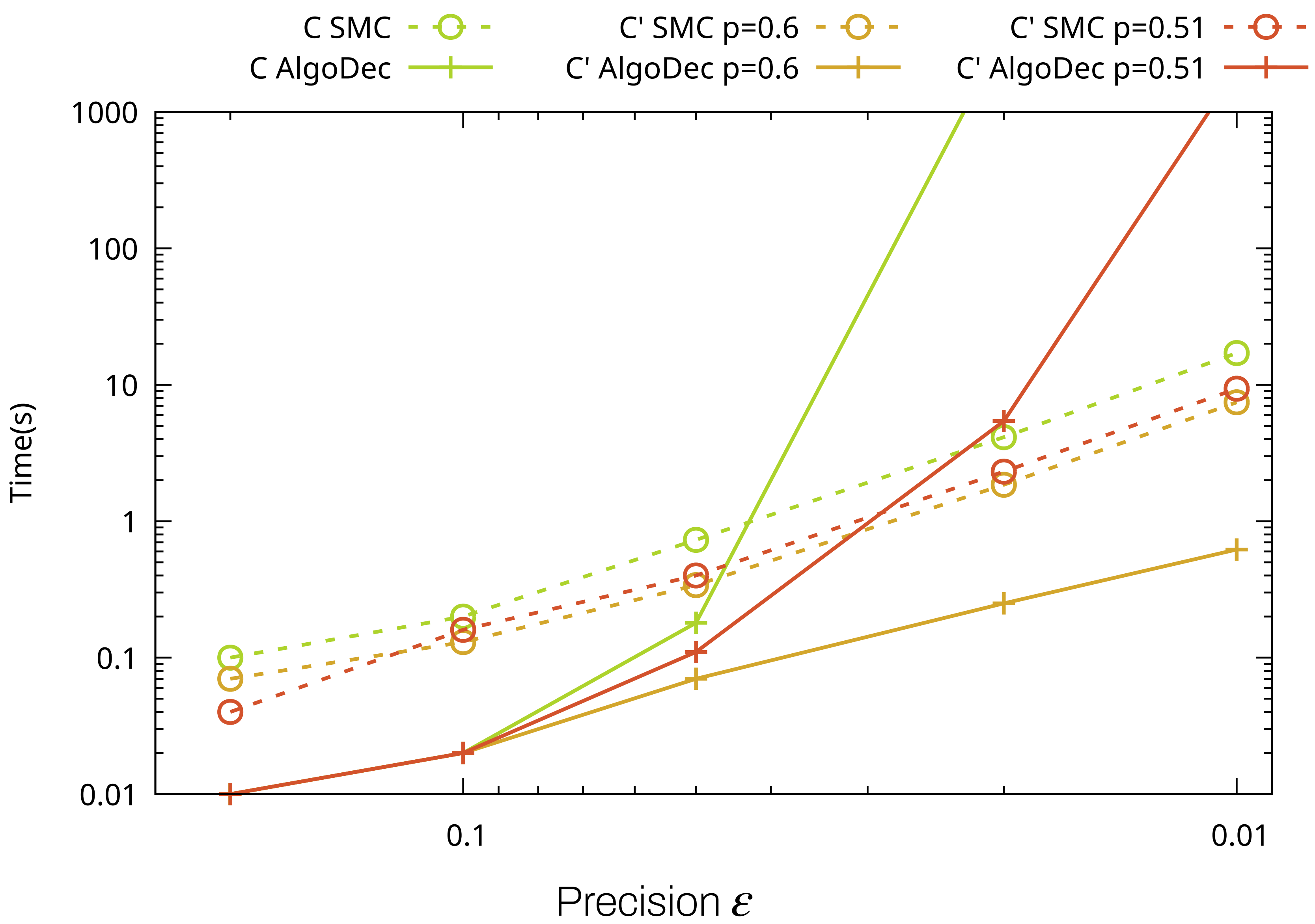


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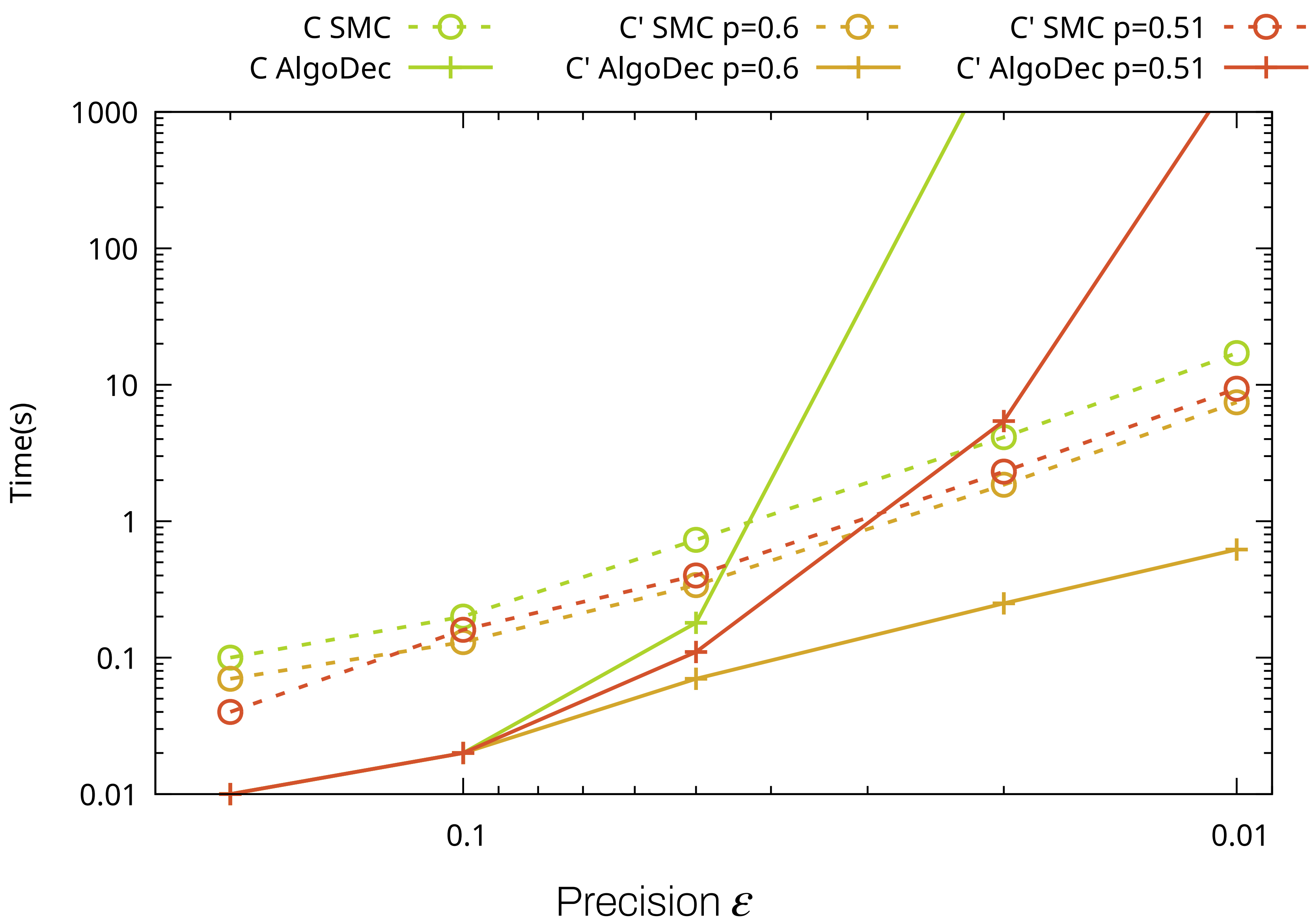


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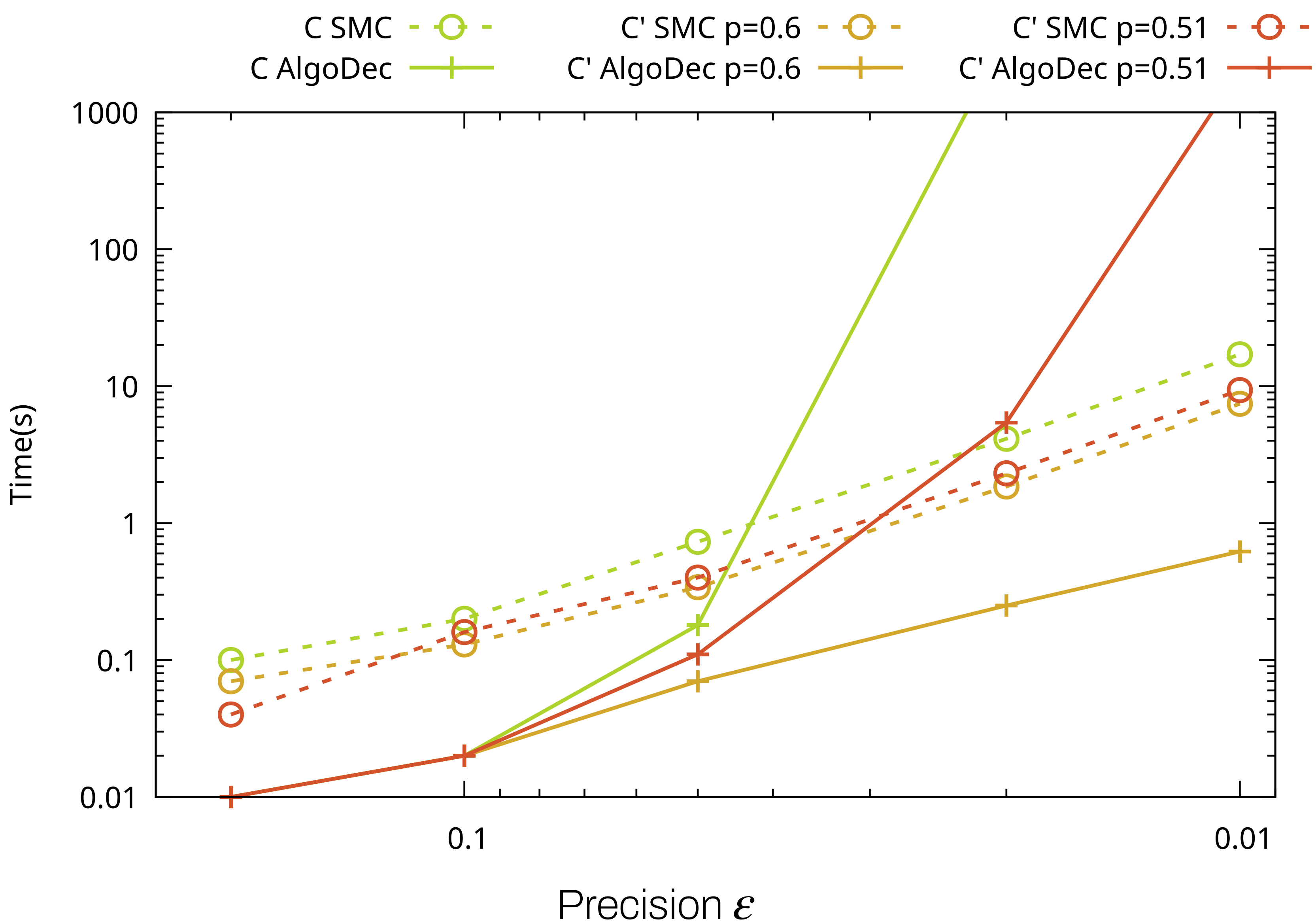


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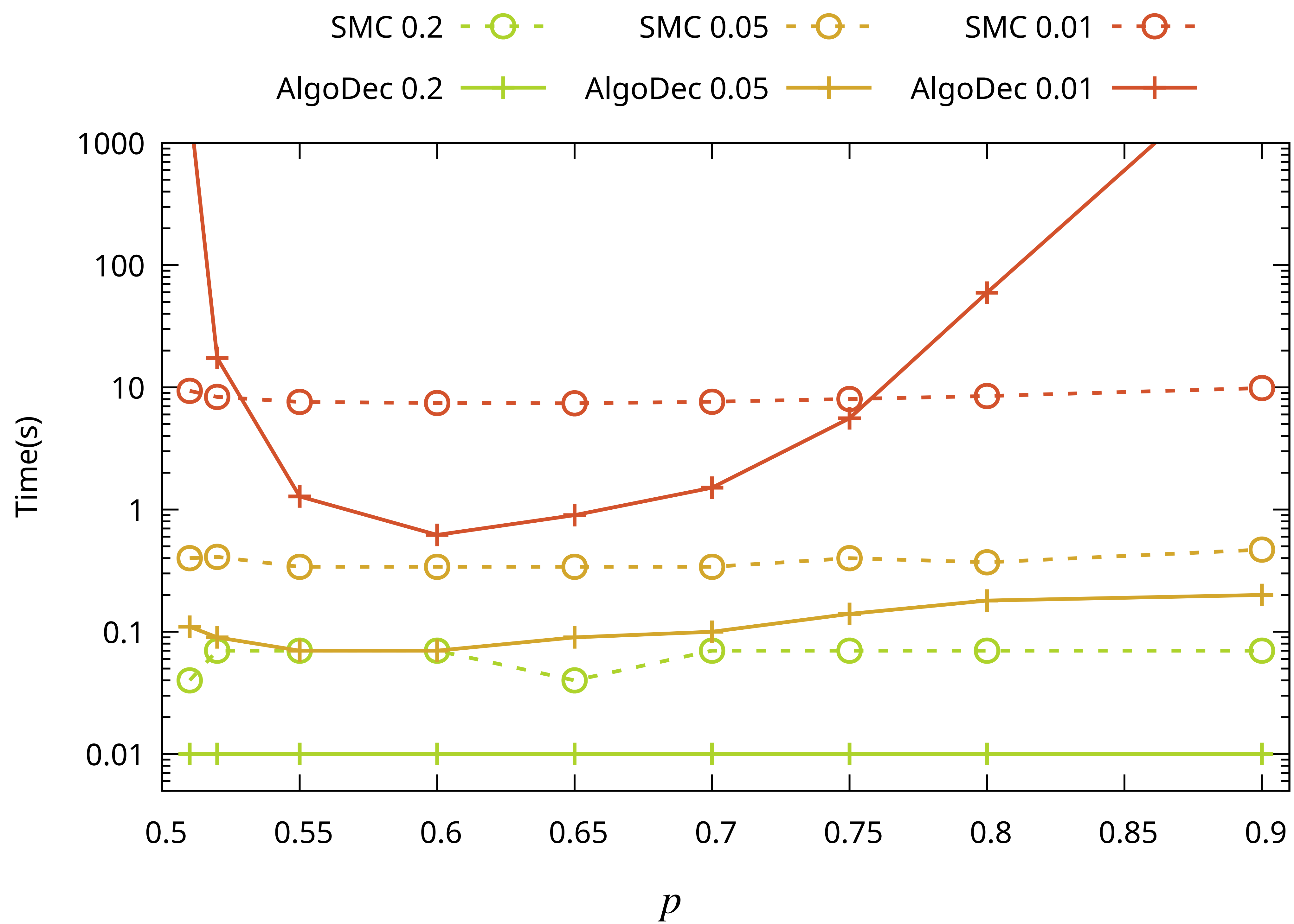
- For that best $p$, Approx behaves very well!

# First example — continued



# Second example

- State-free proba. pushdown automaton $\mathscr{C}$
  $A \xrightarrow{1} B \quad A \xrightarrow{1} C \quad B \xrightarrow{10} \varepsilon \quad B \xrightarrow{10+n} AA$
  $C \xrightarrow{10} A \quad C \xrightarrow{10+n} BB$
- Start from $A$, and target the empty stack

> - It is not decisive
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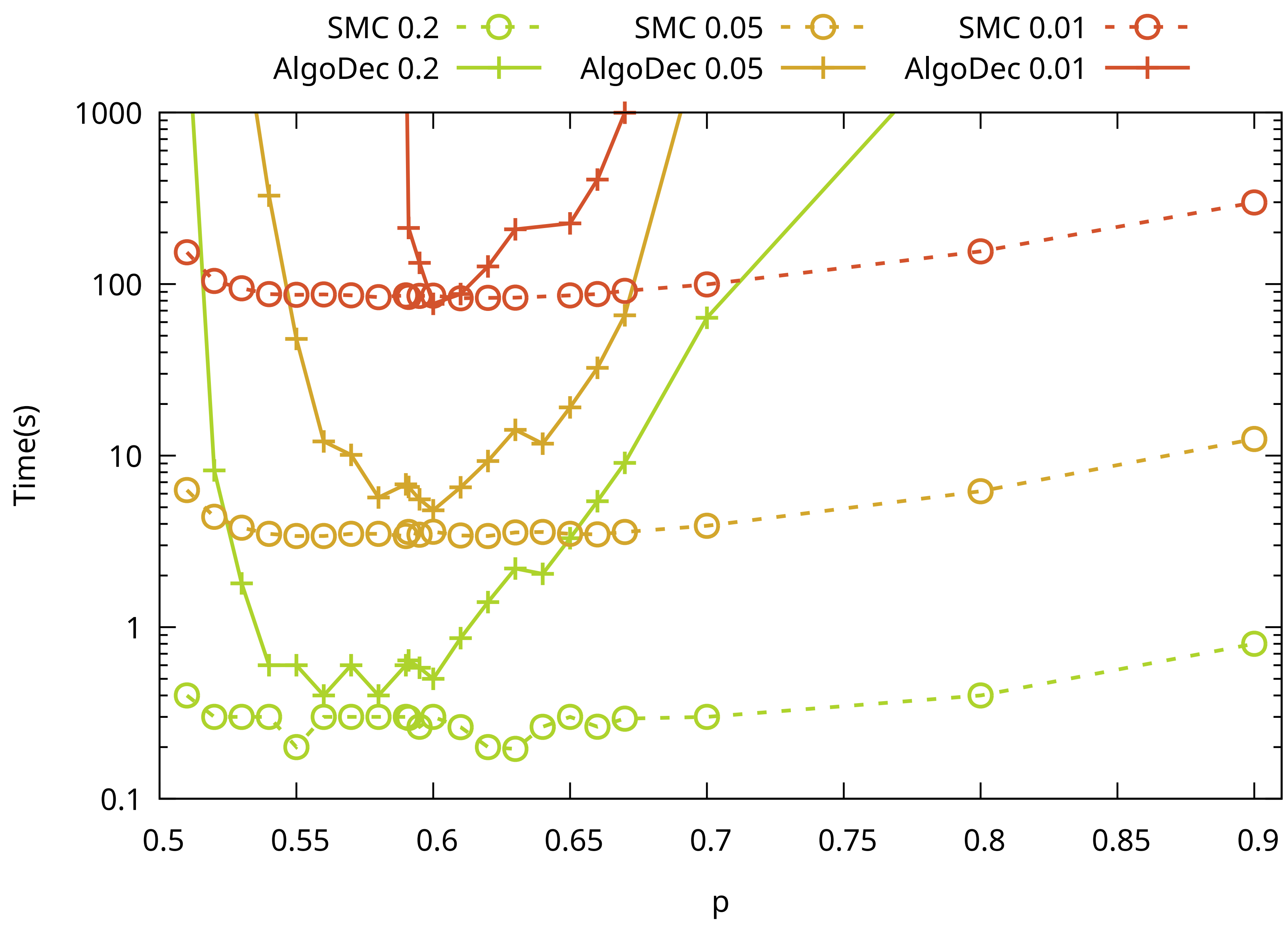

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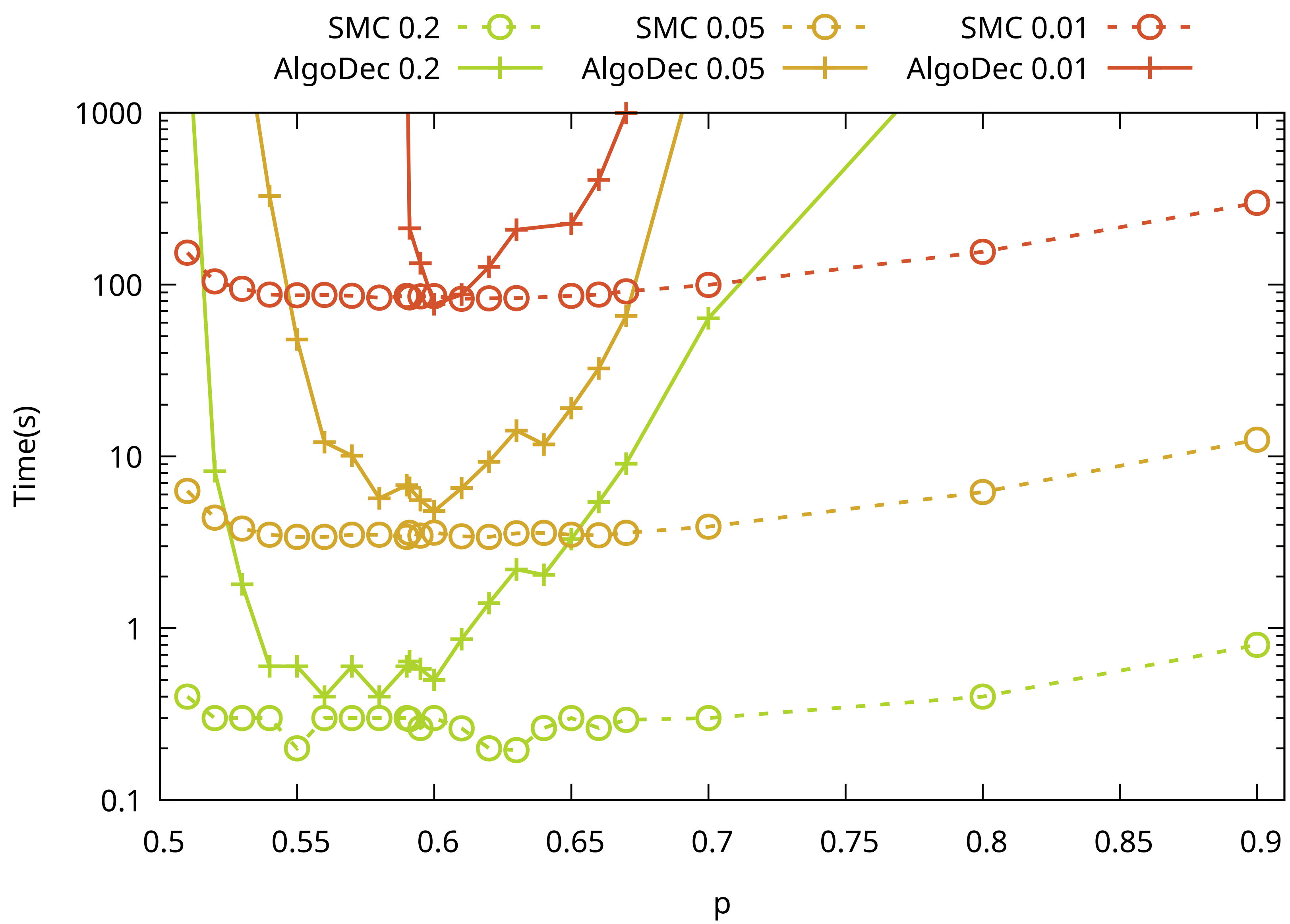


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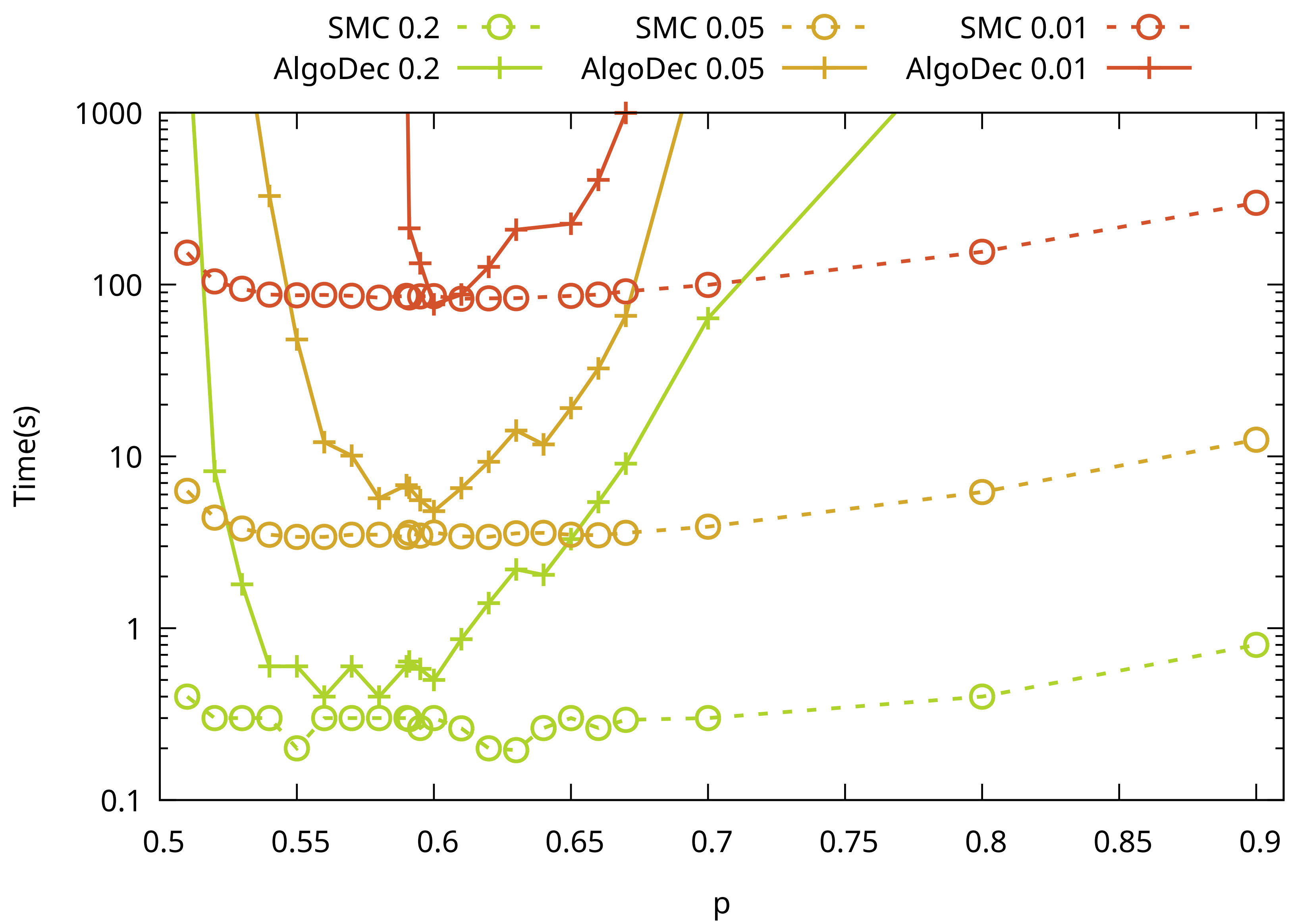


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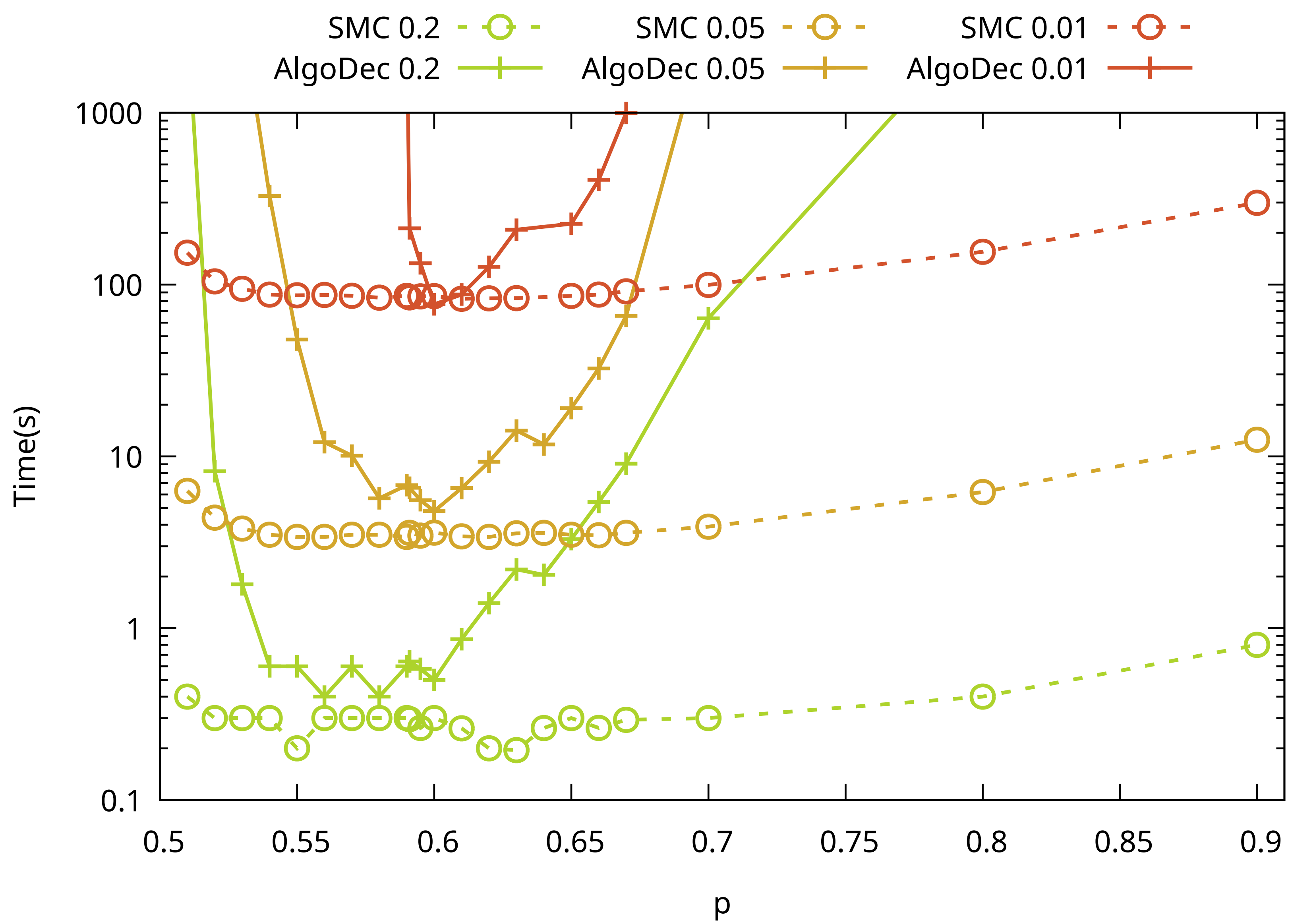


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Statistical guarantees

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