The true colors of memory: A tour of chromatic-memory strategies in zero-sum games on graphs

Patricia Bouyer

Laboratoire Méthodes Formelles
Université Paris-Saclay, CNRS, ENS Paris-Saclay
France

Line of works developed together with Mickael Randour and Pierre Vandenhove.
Some works are co-authors with other people: Antonio Casares, Nathanaël Fijalkow, Stéphane Le Roux, Youssouf Oualhadj.

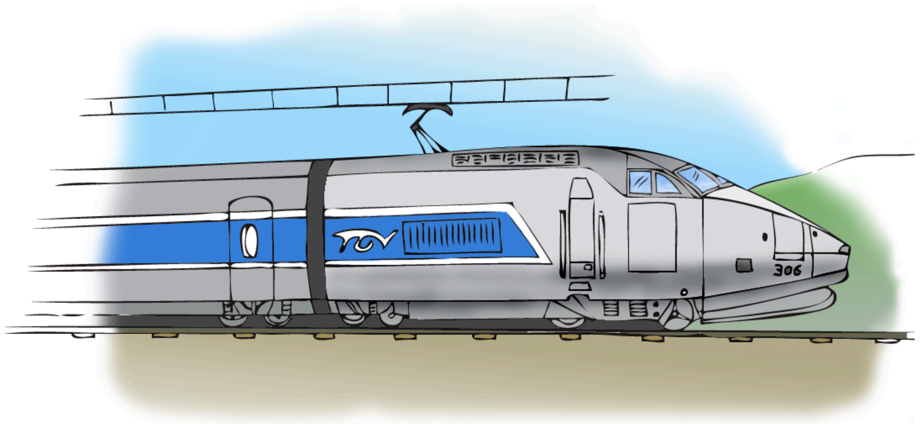
Motivation

—

The setting

Model-checking

System



Model-checking

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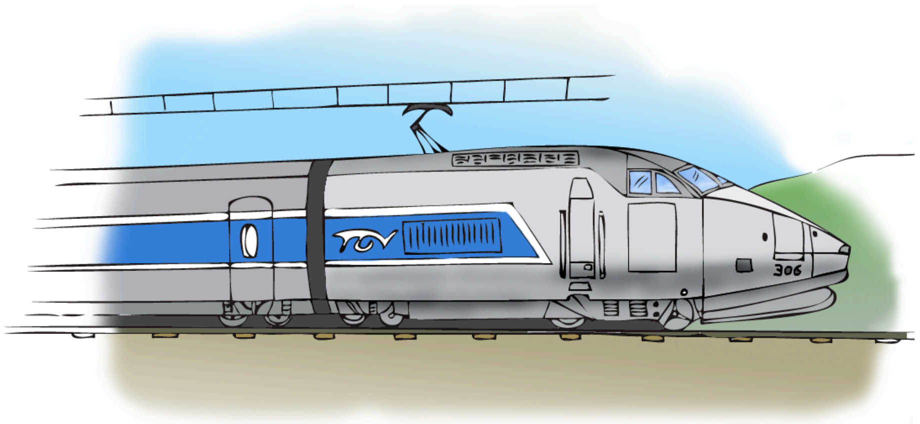

Properties

Model-checking

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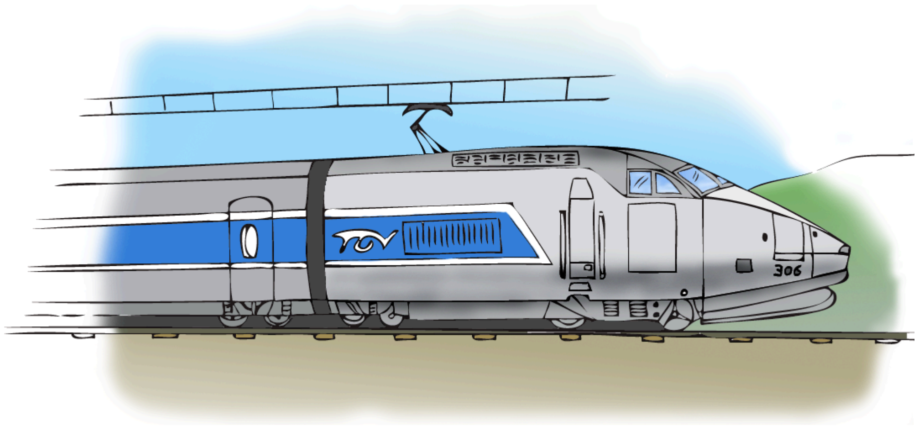

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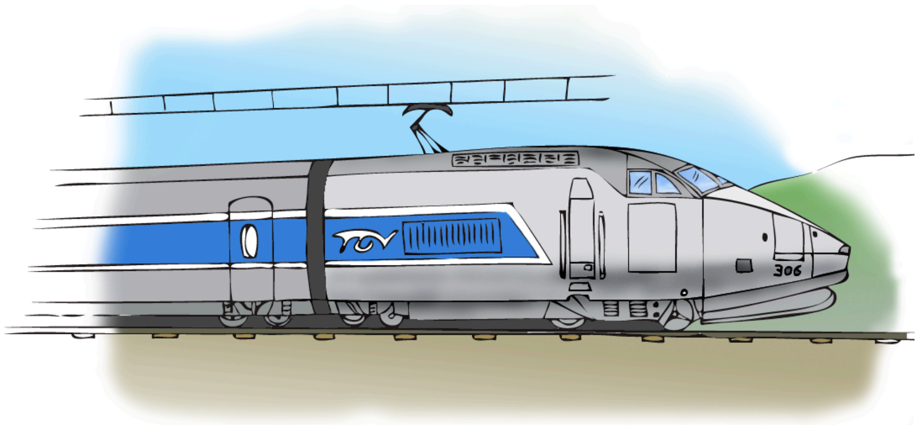

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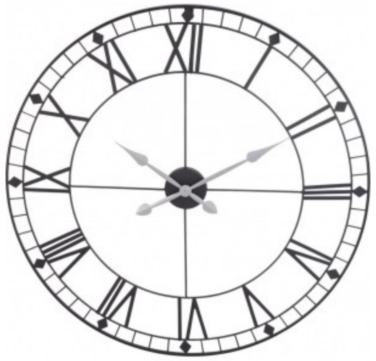

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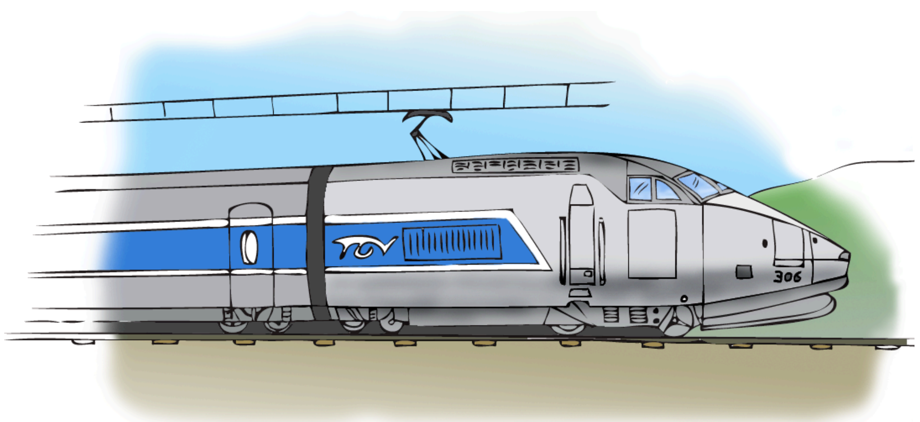

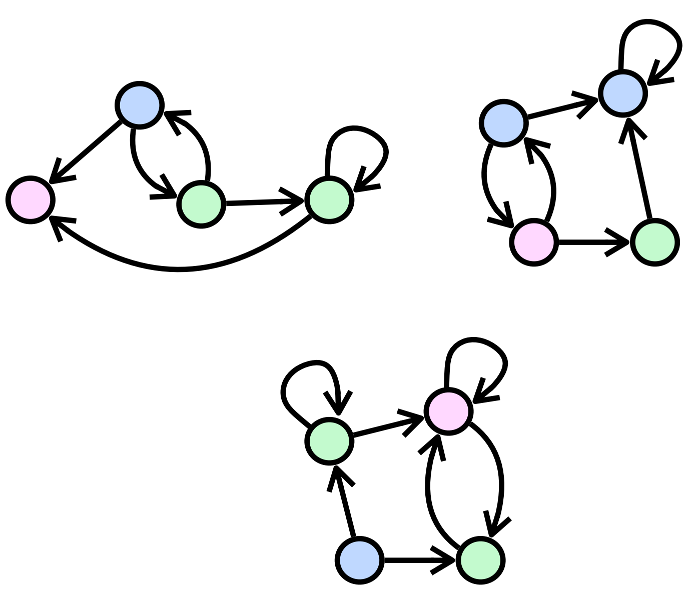

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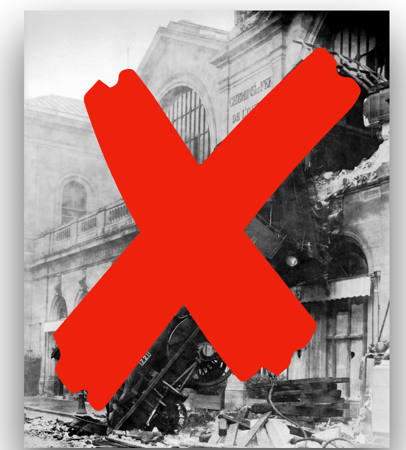

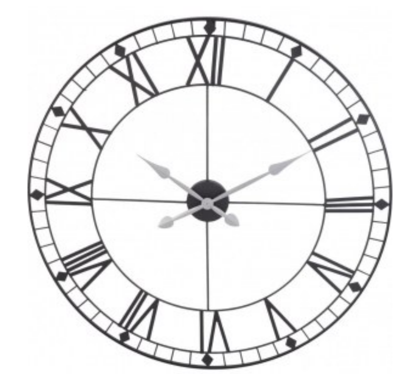

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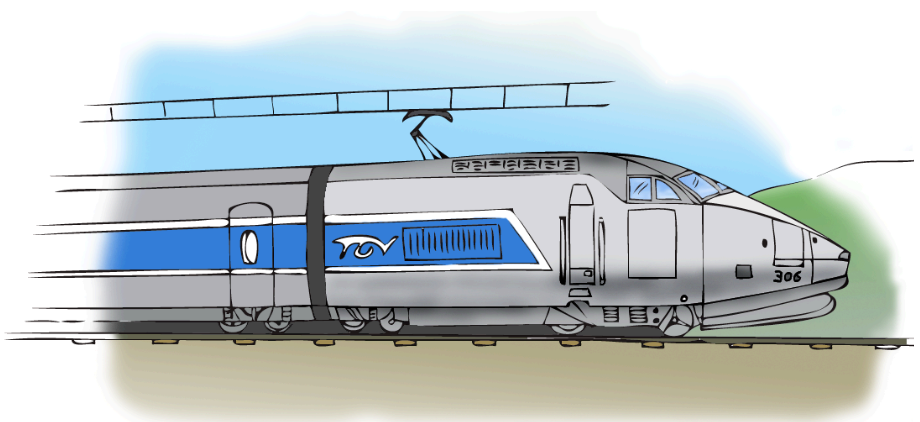
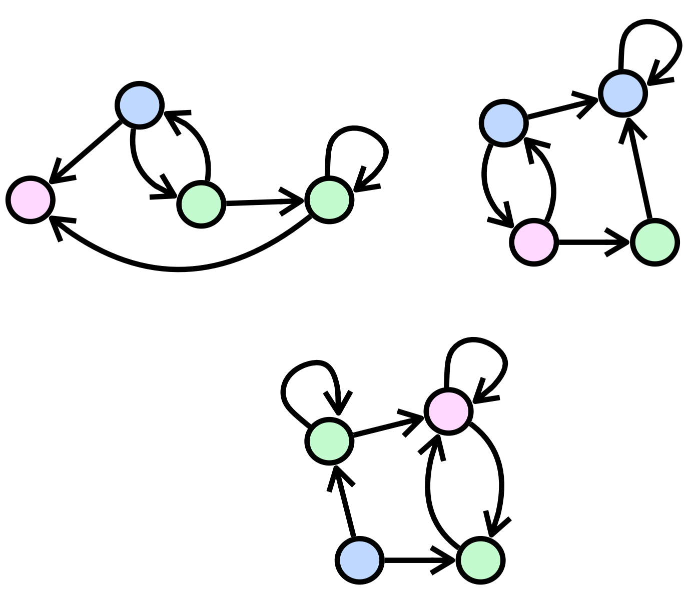


⇩



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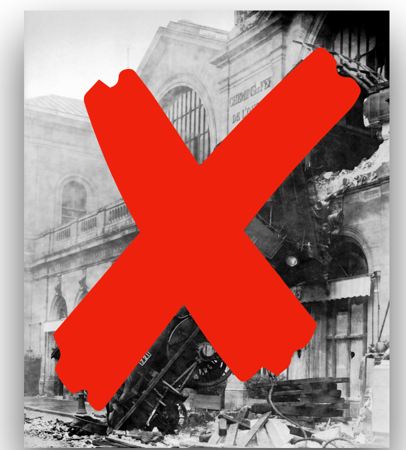

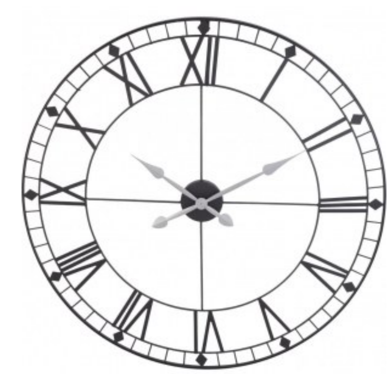

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$$\varphi = \mathbf{AG}\,\neg\text{crash} \wedge \left(\mathbb{P}(\mathbf{F}_{\leq 2\text{h}}\text{arr}) \geq 0{,}9\right)$$

Model-checking

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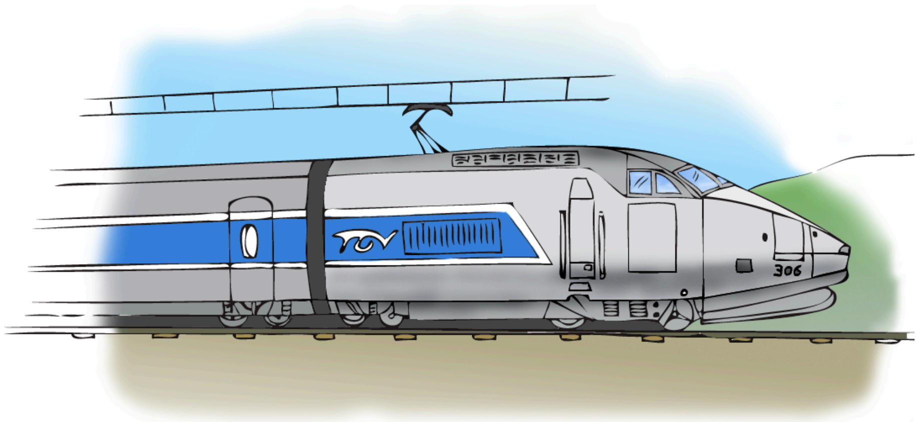

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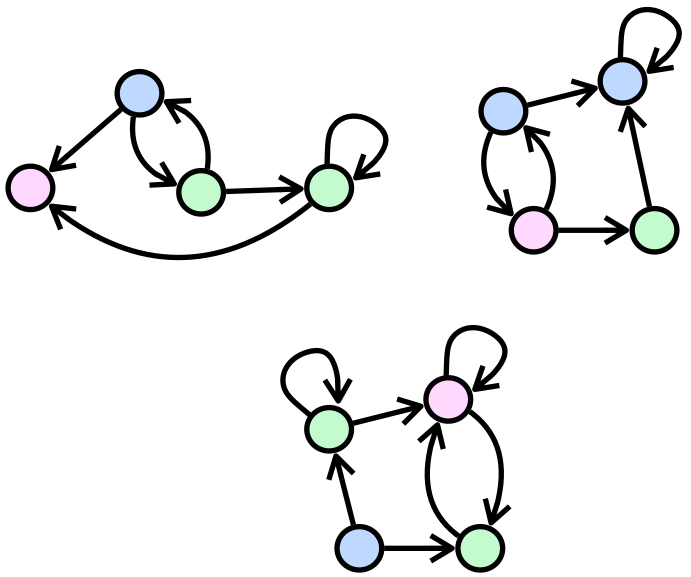

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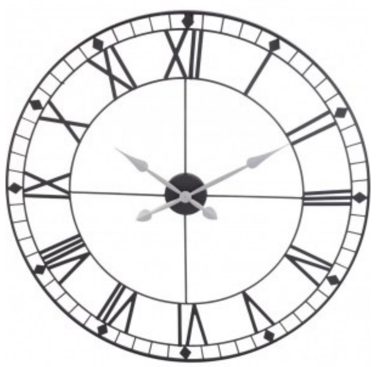

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Model-checking algorithm

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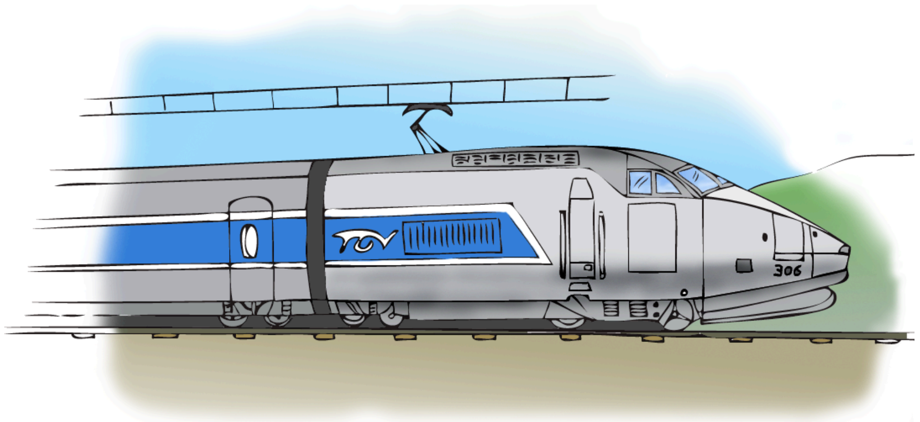

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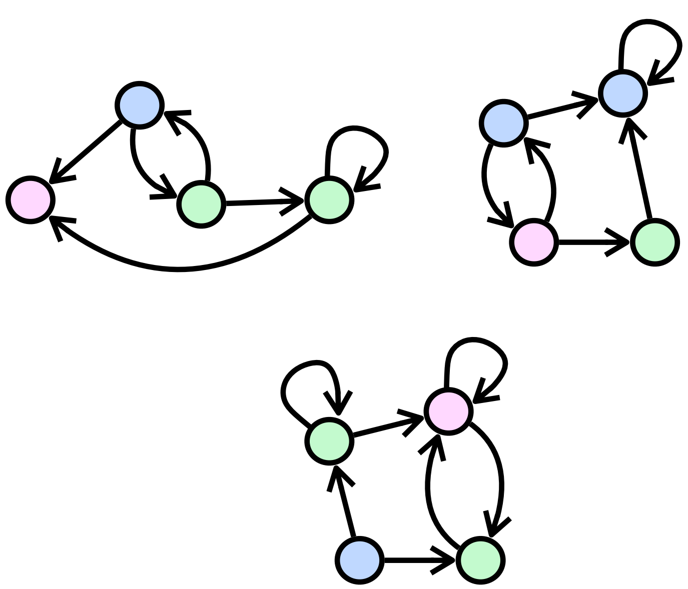

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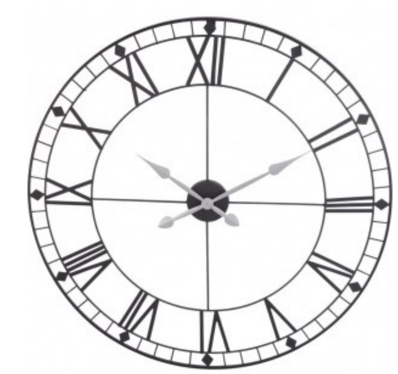

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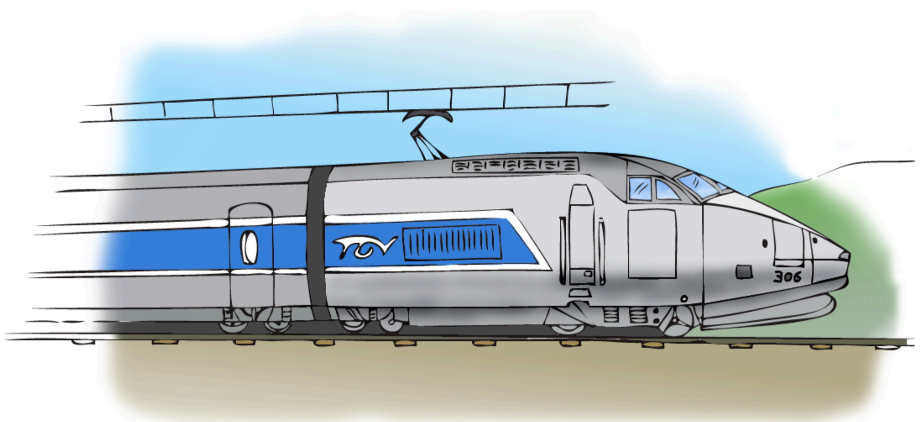
Model-checking algorithm

↓

Yes/No/Why?

Control or synthesis

System



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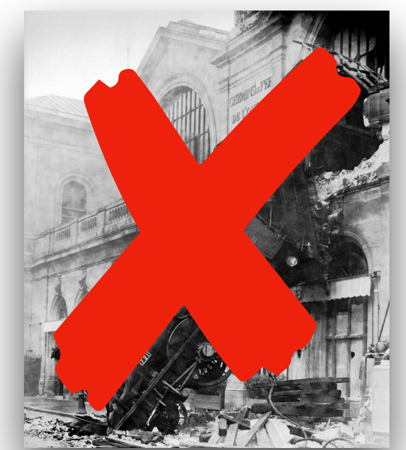

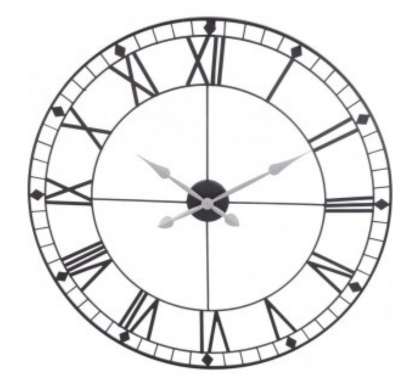

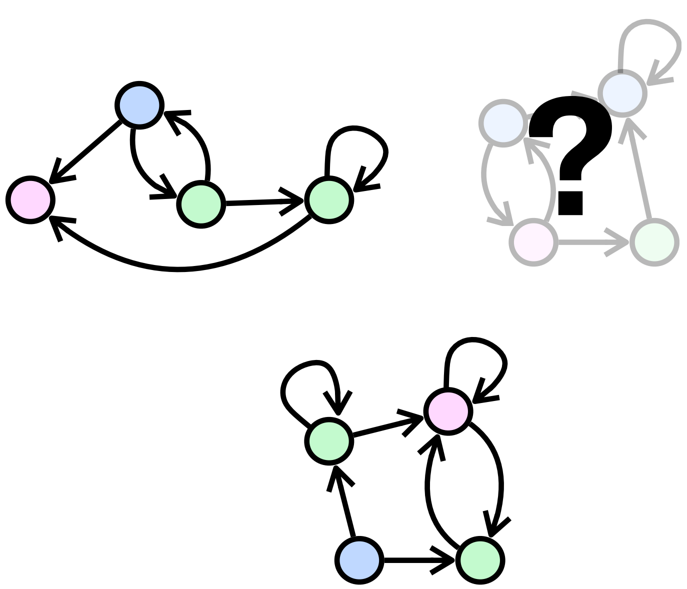

Control/synthesis algorithm

No/Yes/How?

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The talk in one slide

Strategy synthesis for two-player games

Find good and simple controllers for systems interacting with an antagonistic environment

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Minimal information for deciding the next steps

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When are simple strategies sufficient to play optimally?

Our general approach

[Tho95] On the synthesis of strategies in infinite games (STACS'95).
[Tho02] Thomas. Infinite games and verification (CAV'02).
[GU08] Grädel, Ummels. Solution concepts and algorithms for infinite multiplayer games (New Perspectives in Games and Interactions, 2008).
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- Use **graph-based game models** (state machines) to represent the system and its evolution

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Our general approach

- Use **graph-based game models** (state machines) to represent the system and its evolution
- Use **game theory concepts** to express admissible situations
  - Winning strategies
  - (Pareto-)Optimal strategies
  - Nash equilibria
  - Subgame-perfect equilibria
  - ...

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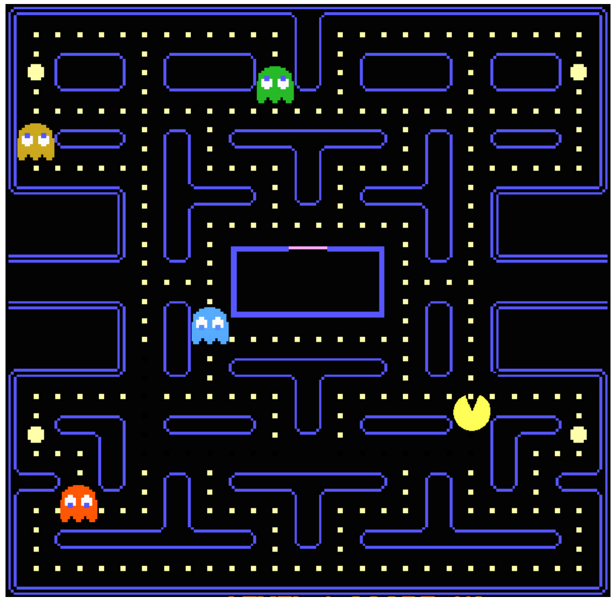
Games
What they often are



Games
A broader sense

Interaction

Goal

- Model and analyze (using math. tools) situations of interactive decision making

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Ingredients

- Several decision makers (players)
- Possibly each with different goals
- The decision of each player impacts the outcome of all

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Wide range of applicability

« [...] it is a context-free mathematical toolbox. »

- Social science: e.g. social choice theory
- Theoretical economics: e.g. models of markets, auctions
- Political science: e.g. fair division
- Biology: e.g. evolutionary biology
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[MSZ13] Maschler, Solan, Zamir. Game theory (2013).

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+ Computer science

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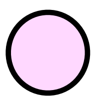
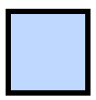
Turn-based games on graphs

States   Edges

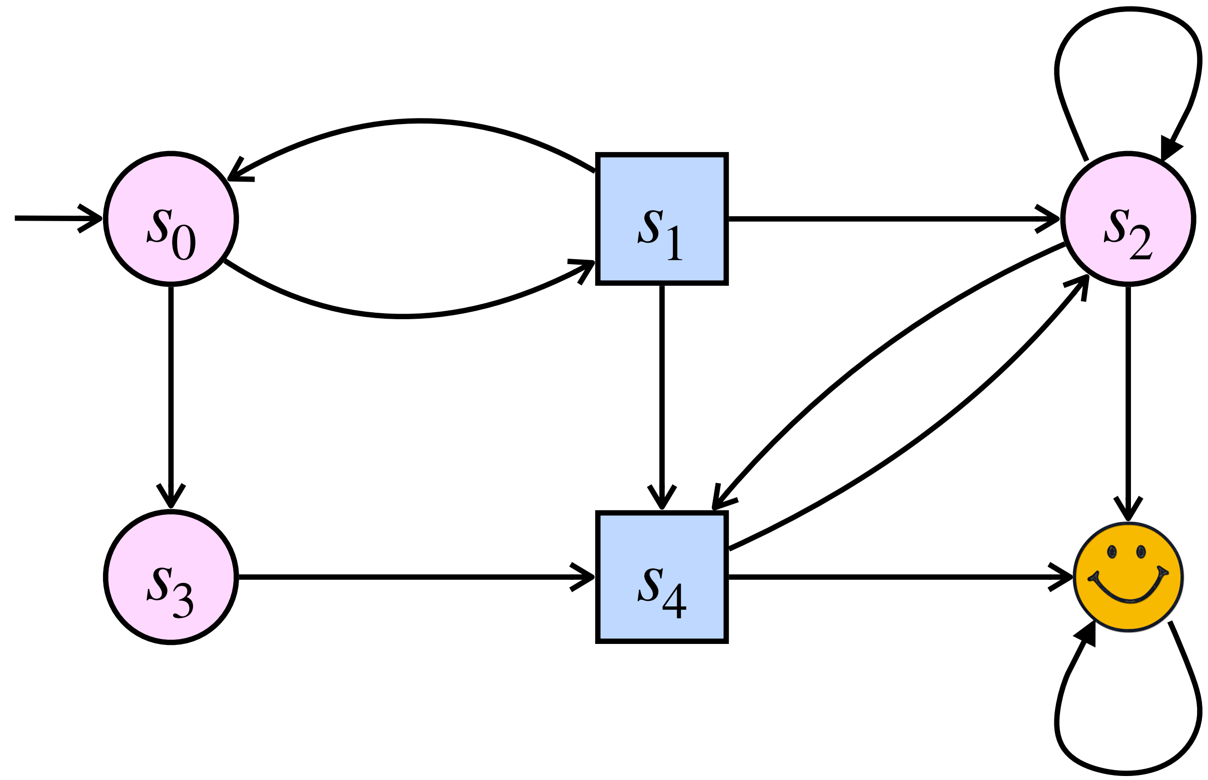
$$\mathcal{G} = (S, s_0, S_1, S_2, E)$$



: player P_1



: player P_2



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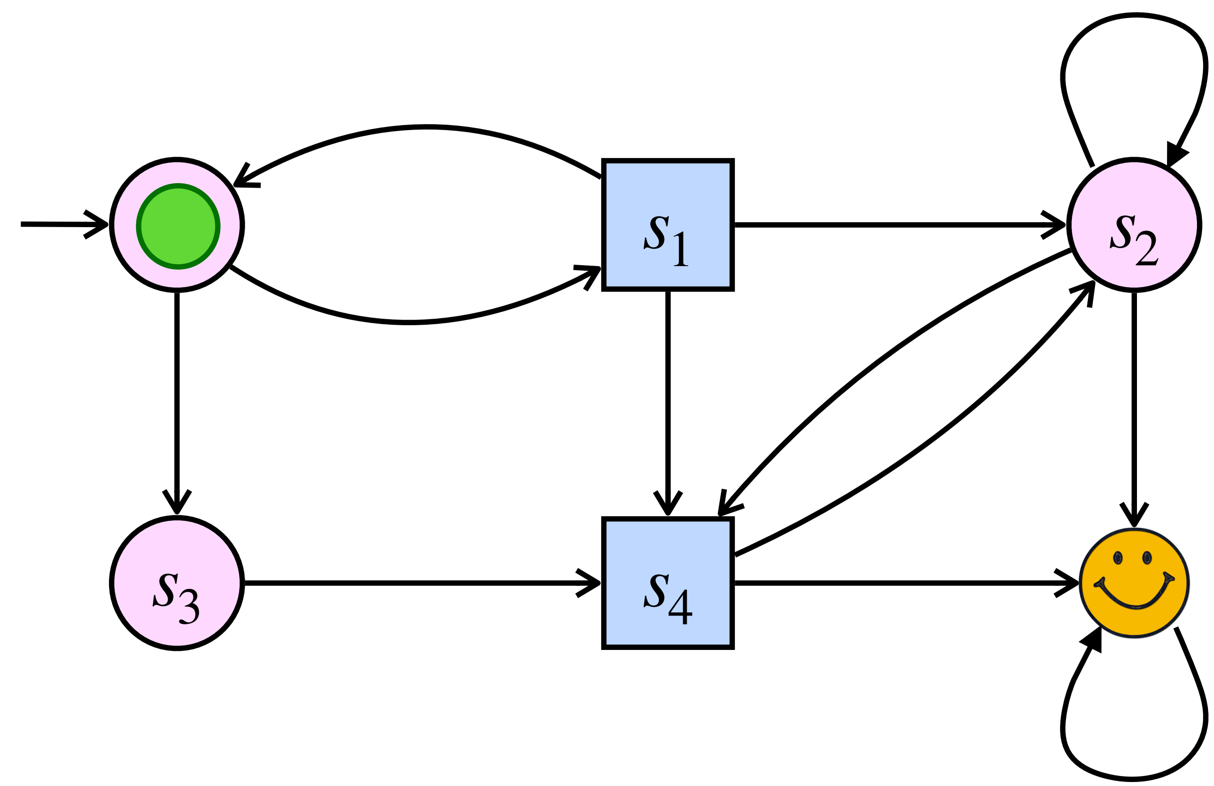

s_0

Turn-based games on graphs

States — Edges

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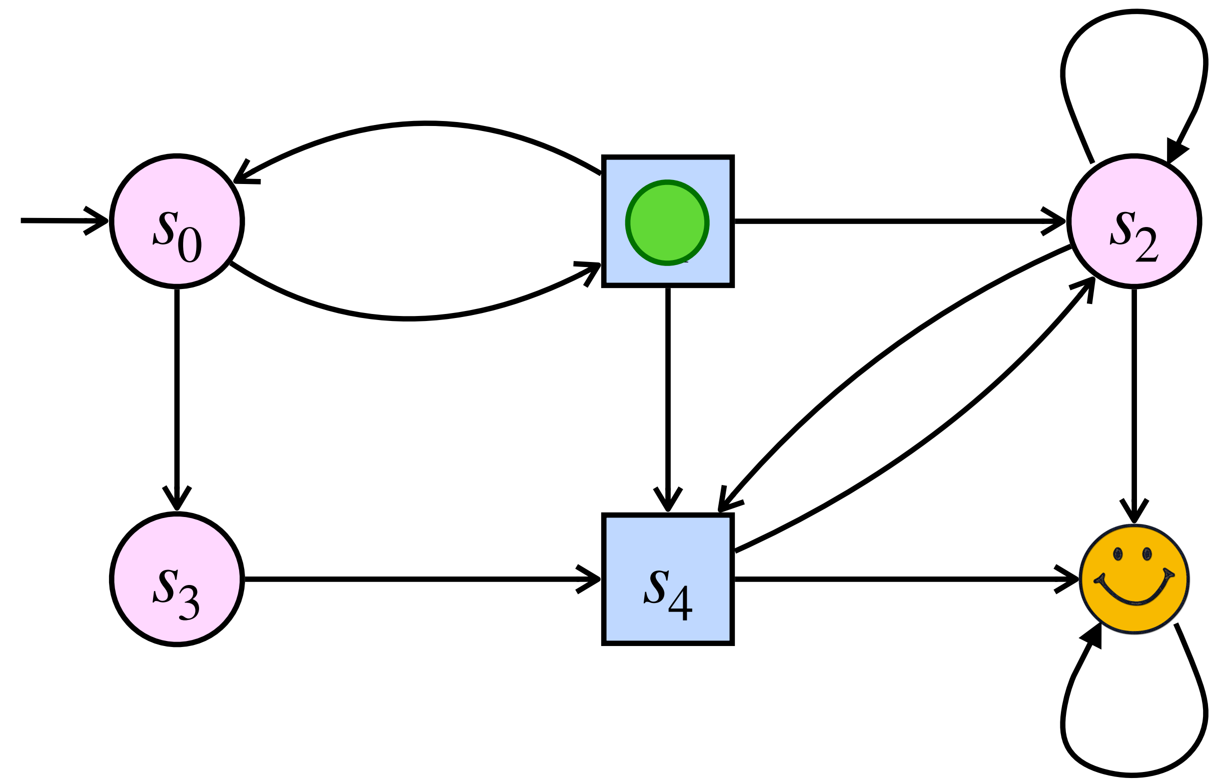
□ : player P_2



$$s_0 \rightarrow s_1$$

1. P_1 chooses the edge (s_0, s_1)

Turn-based games on graphs

States → S, Edges → E

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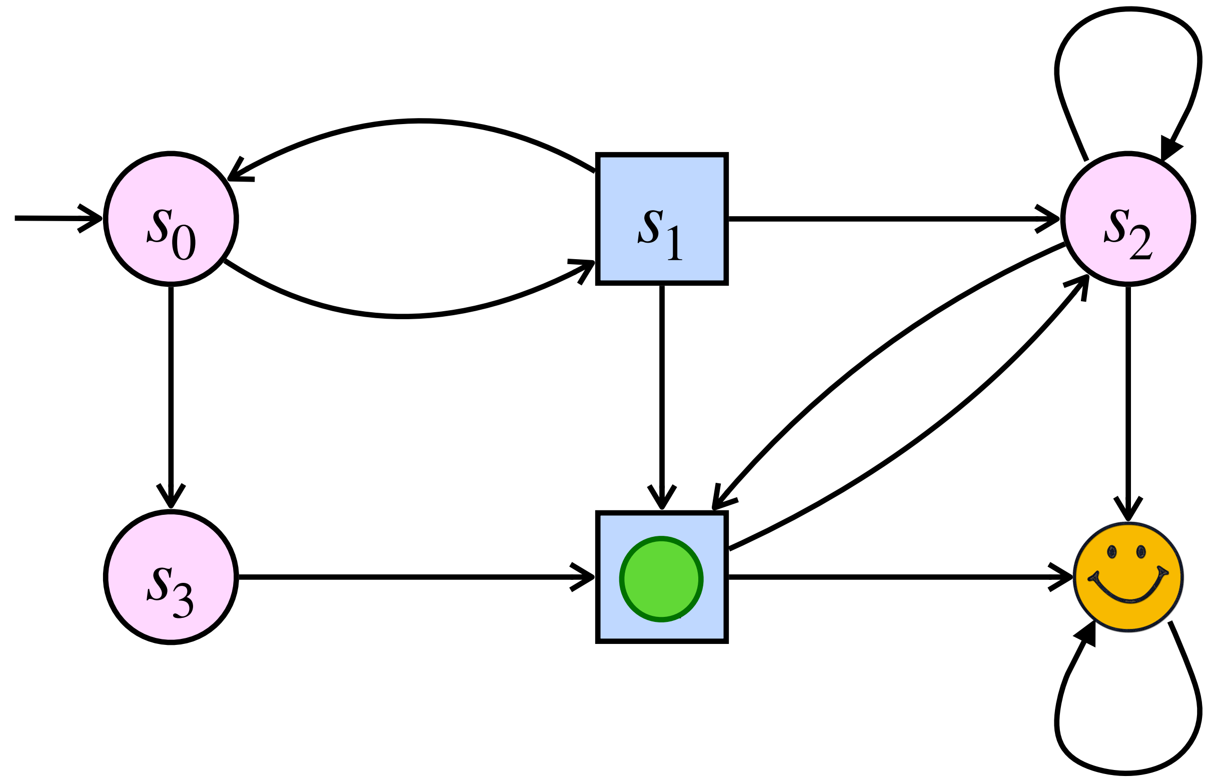

○ : player P_1

□ : player P_2

$$s_0 \rightarrow s_1 \rightarrow s_4$$

1. P_1 chooses the edge (s_0, s_1)
2. P_2 chooses the edge (s_1, s_4)

Turn-based games on graphs

States → S, Edges → E

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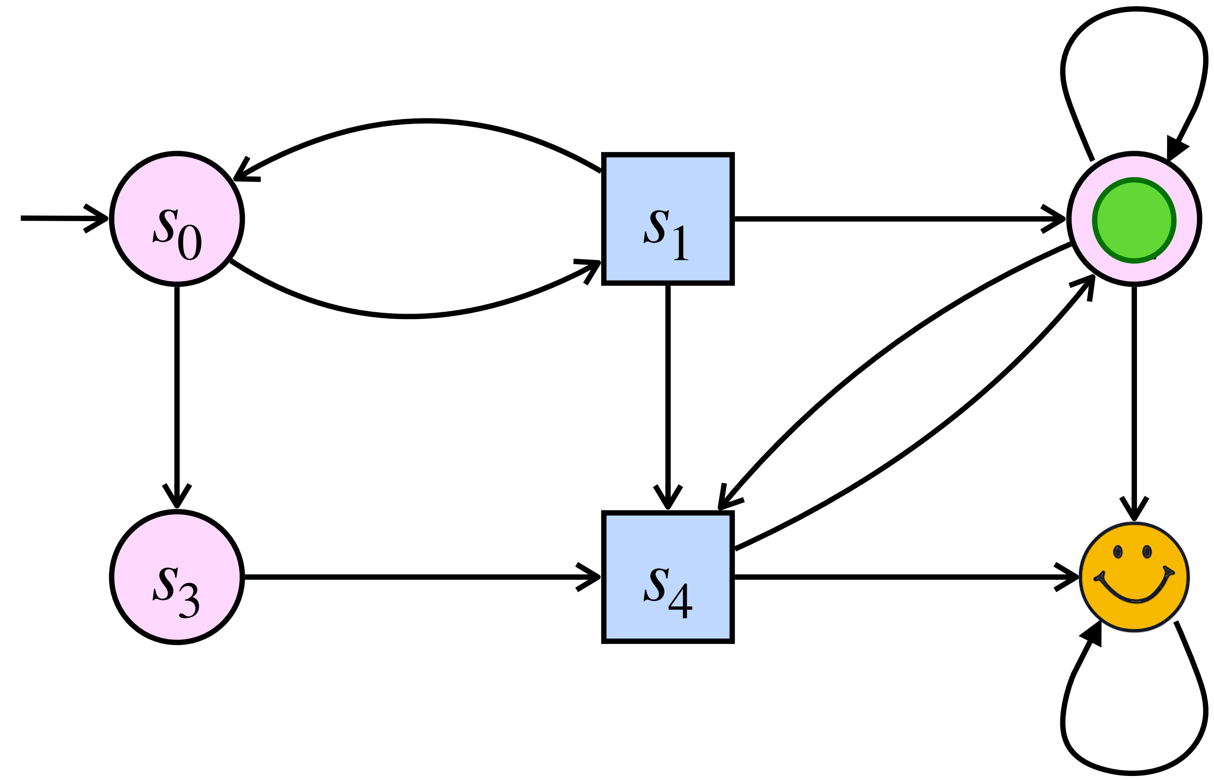

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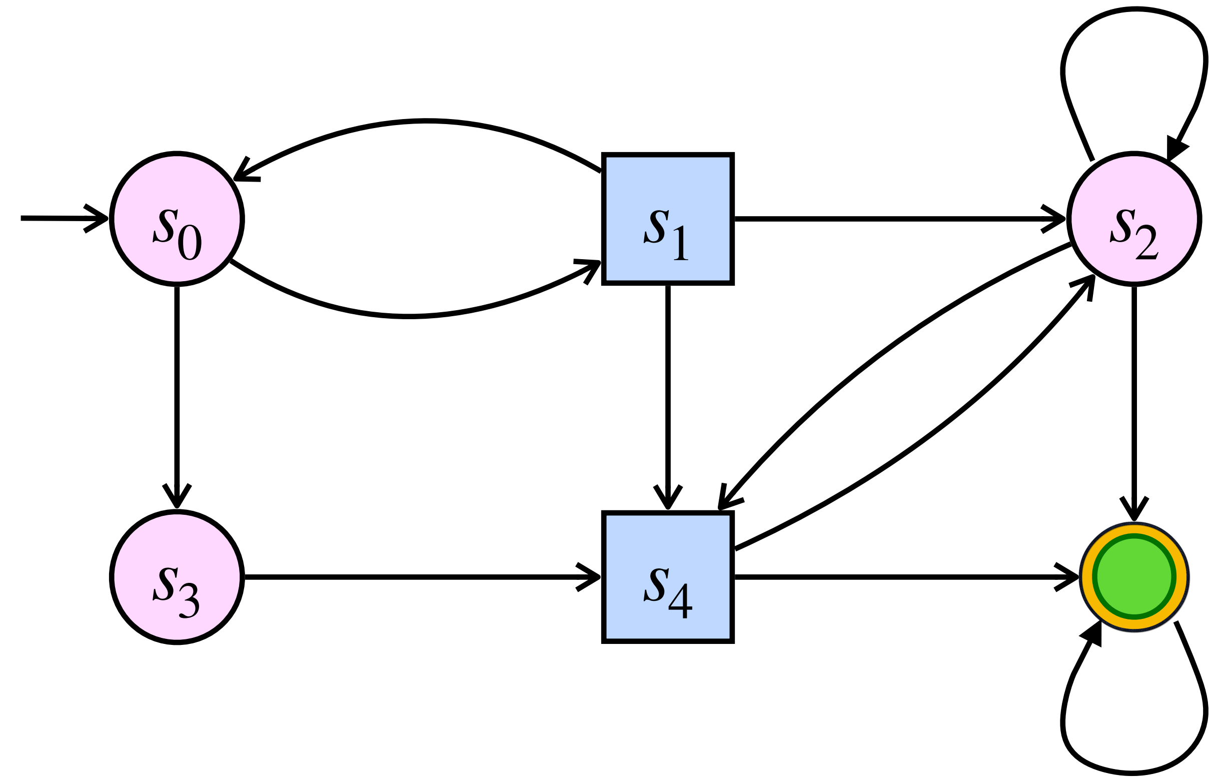

$s_0 \to s_1 \to s_4 \to s_2 \to$ 

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3. P_2 chooses the edge (s_4, s_2)
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Turn-based games on graphs

States → S, Edges → E

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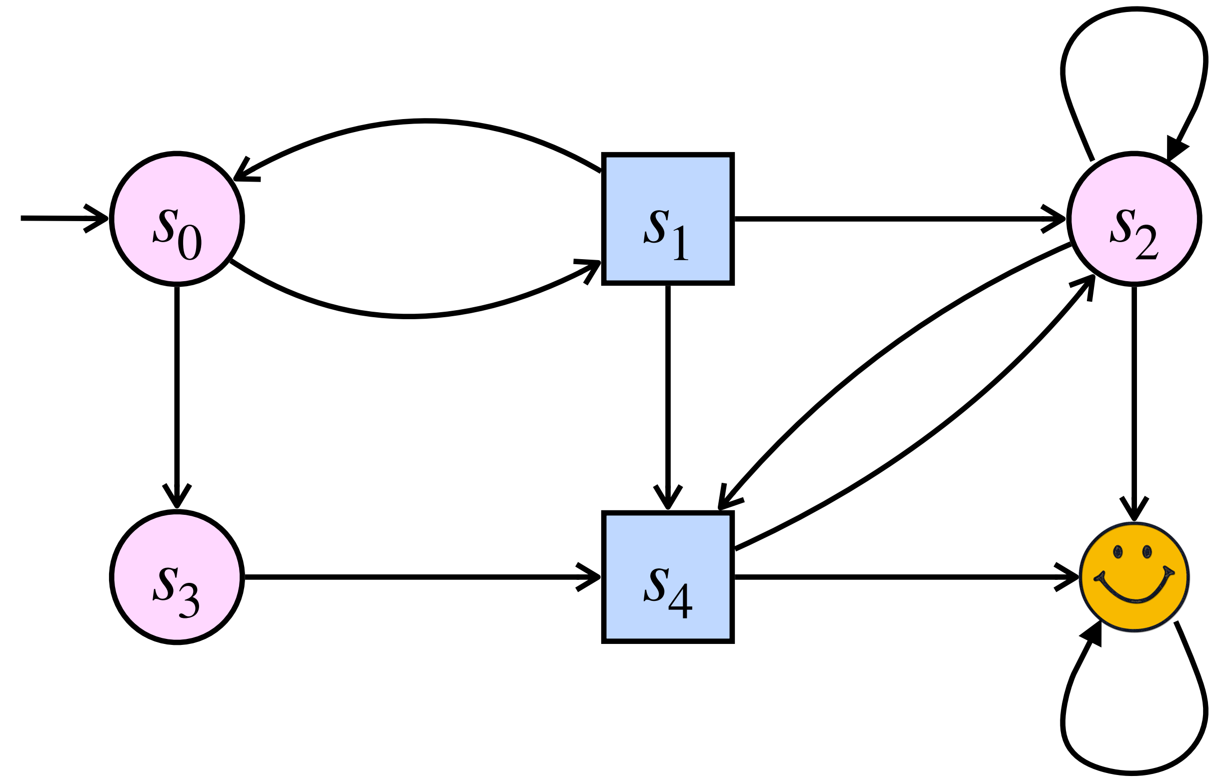

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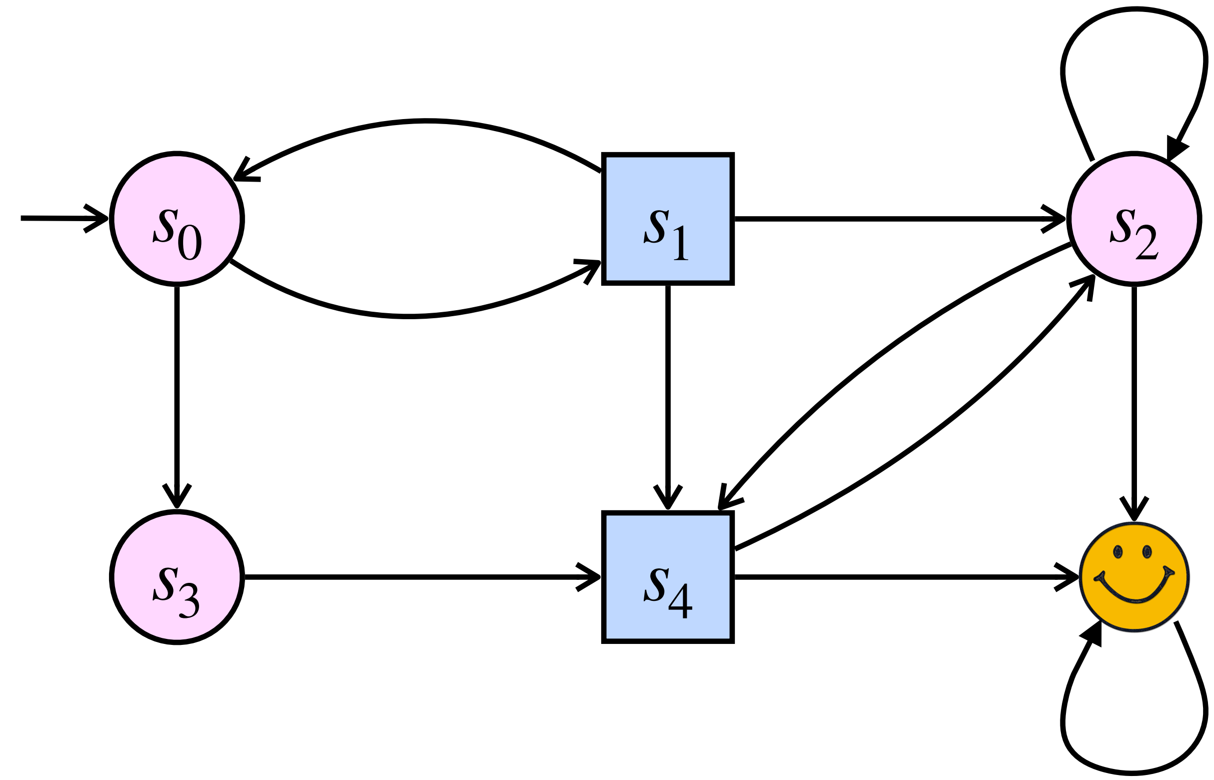

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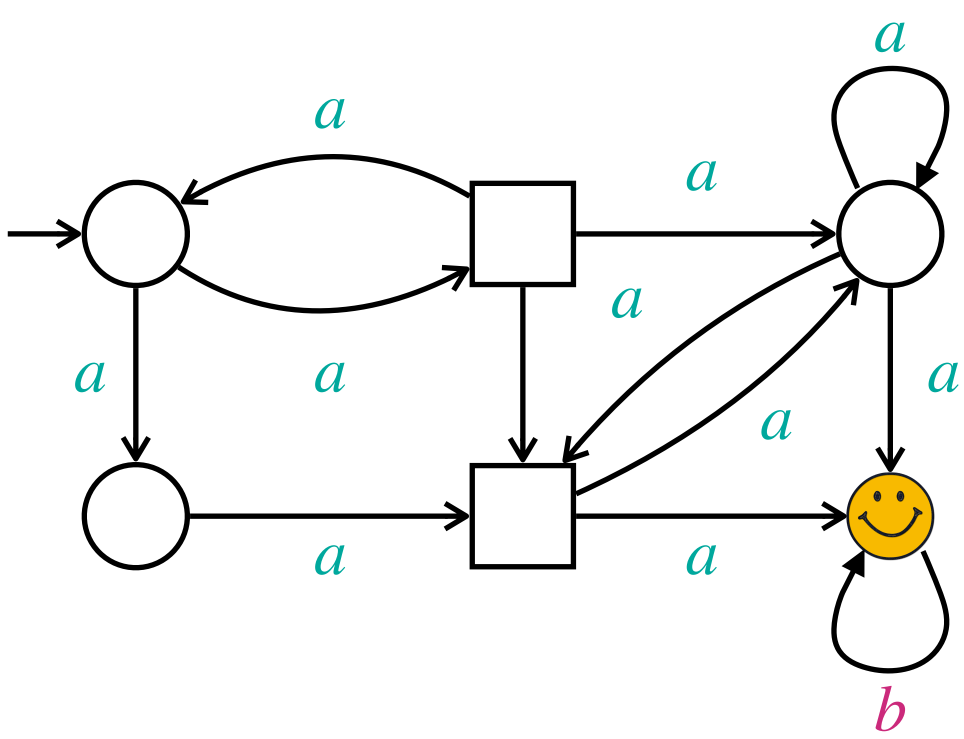
Players use **strategies** to play.
A strategy for P_i is $\sigma_i : S^*S_i \to E$

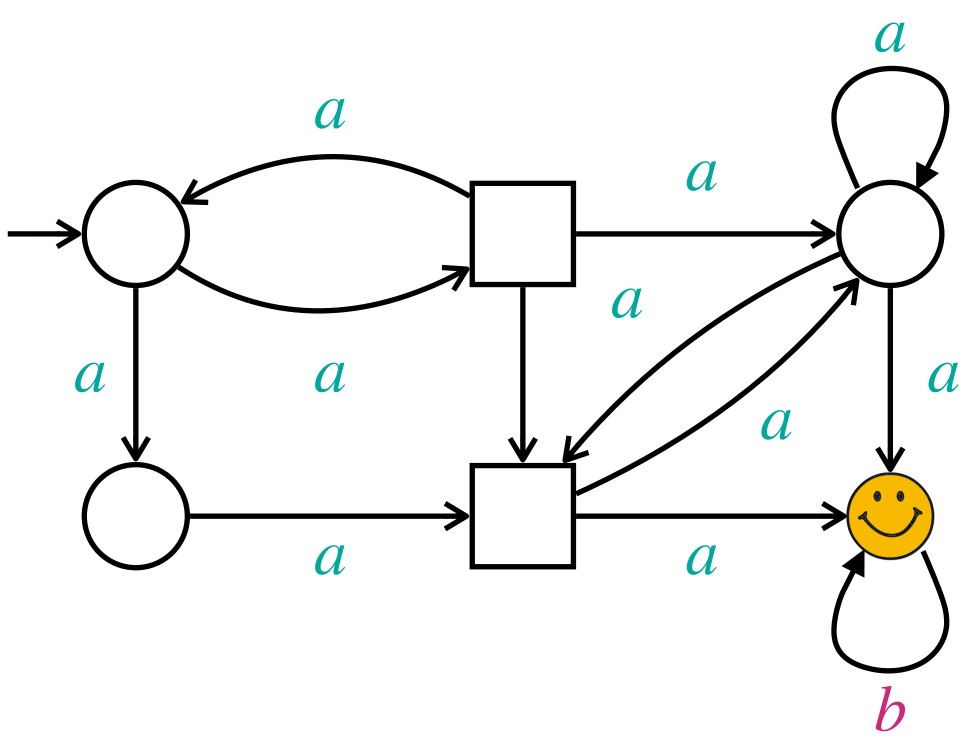
Objectives for the players



$$C = \{a, b\}$$ set of colors

$$E \subseteq S \times C \times S$$

Objectives for the players



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- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

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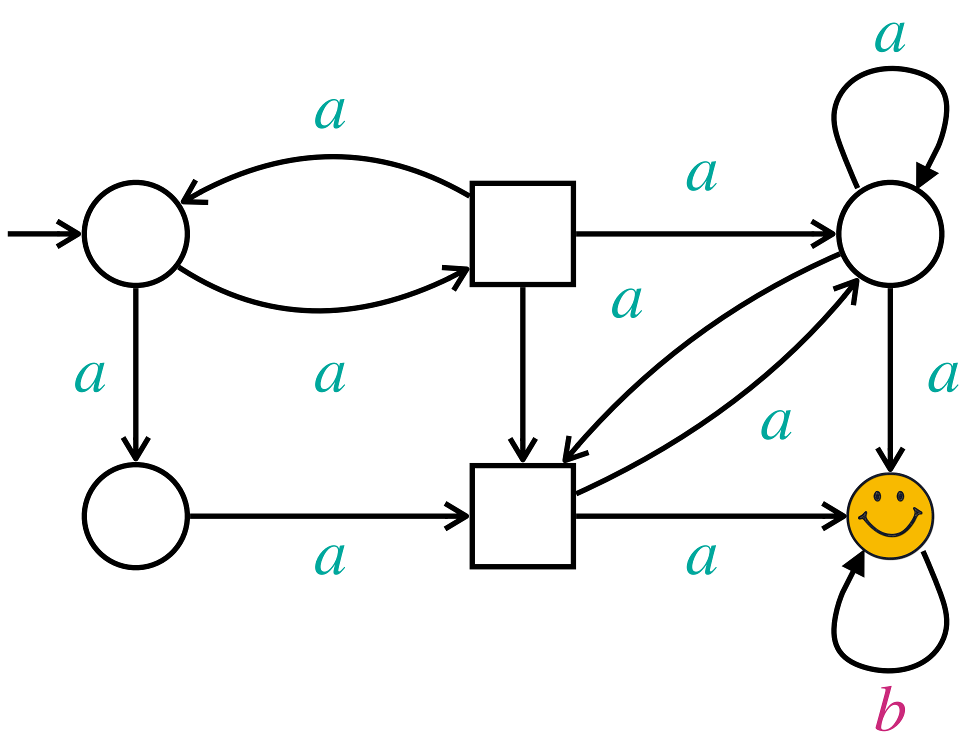

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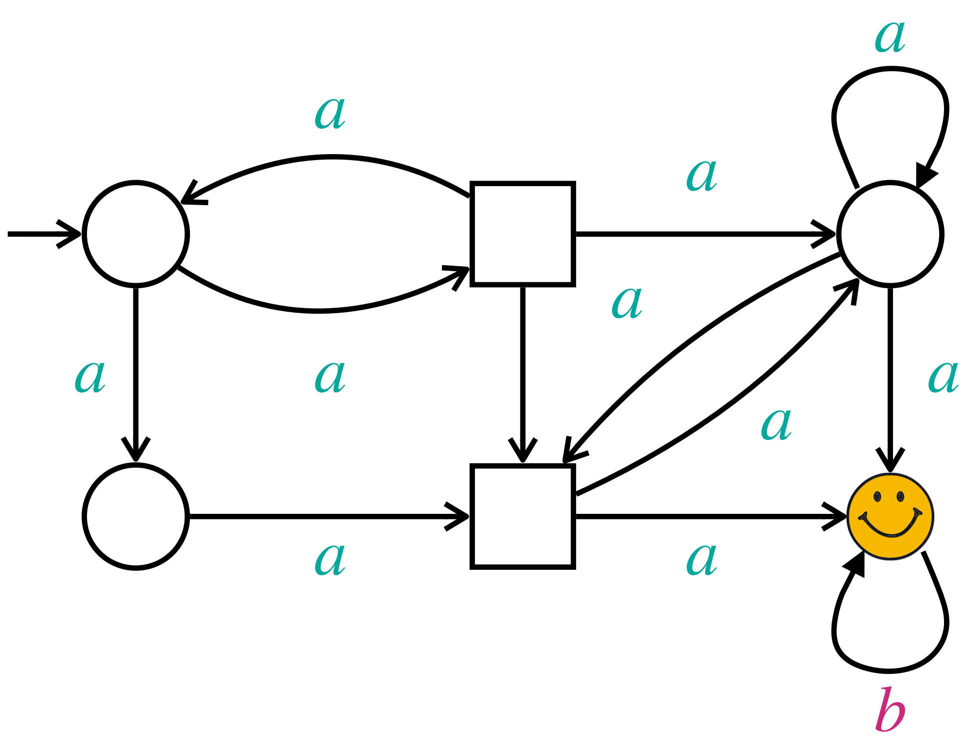

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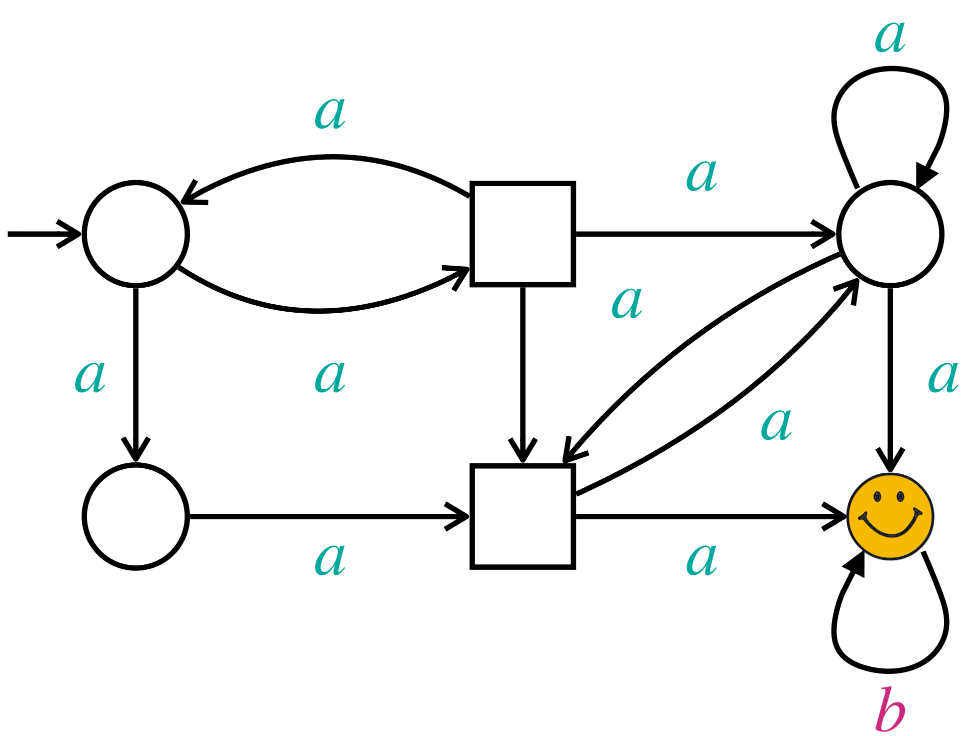
- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

Objectives for the players



Zero-sum assumption

$C = \{a, b\}$ set of colors

$E \subseteq S \times C \times S$

- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

$W_2 = W_1^c$

- Payoff function: $p_i : C^\omega \to \mathbb{R}$, e.g. mean-payoff

$p_1 + p_2 = 0$

- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

$\sqsubseteq_2 = \sqsubseteq_1^{-1}$

Objectives for the players

Zero-sum assumption

$C = \{a, b\}$ set of colors

$E \subseteq S \times C \times S$

- Winning objective for P_i: $W_i \subseteq C^\omega$, e.g. $W_1 = C^* \cdot b \cdot C^\omega$

$W_2 = W_1^c$

- We focus on winning objectives, and write W for W_1

0

- Preference relation: $\sqsubseteq_i \subseteq C^\omega \times C^\omega$ (total preorder)

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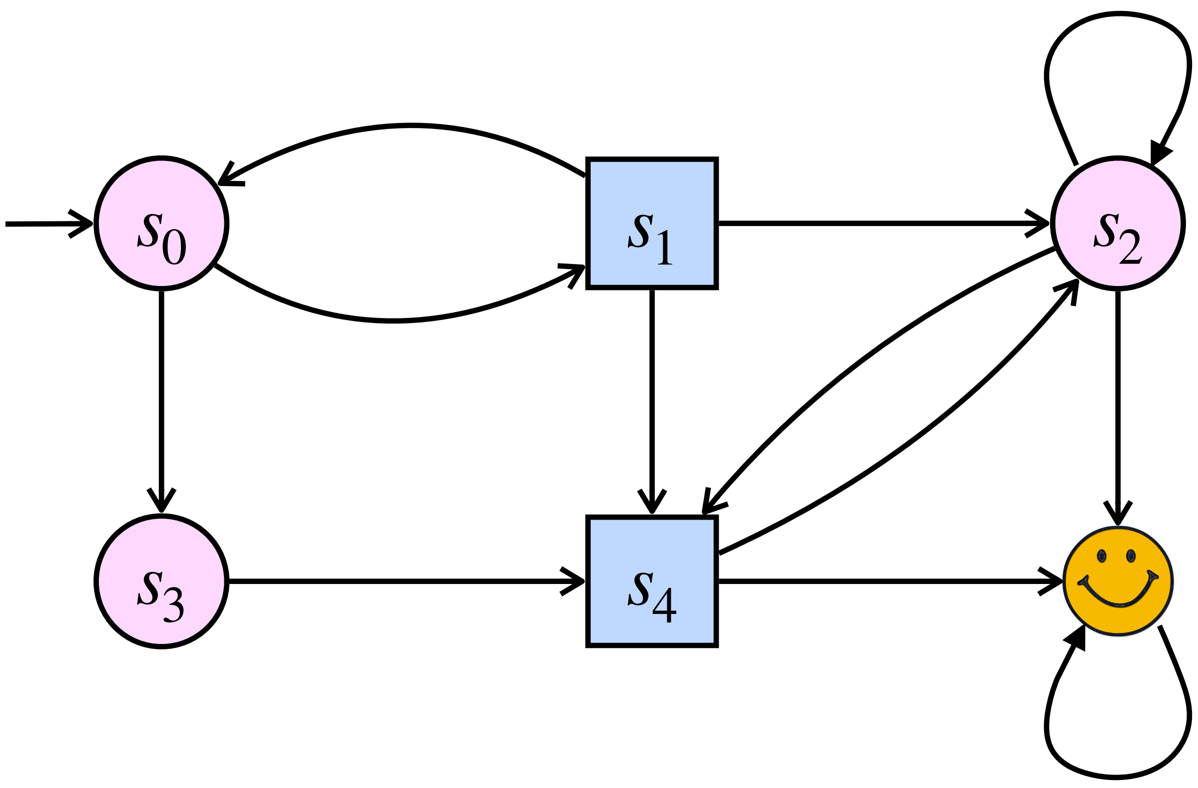
What does it mean to win a game?

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out$(\sigma_i)$.

Outcomes of a strategy



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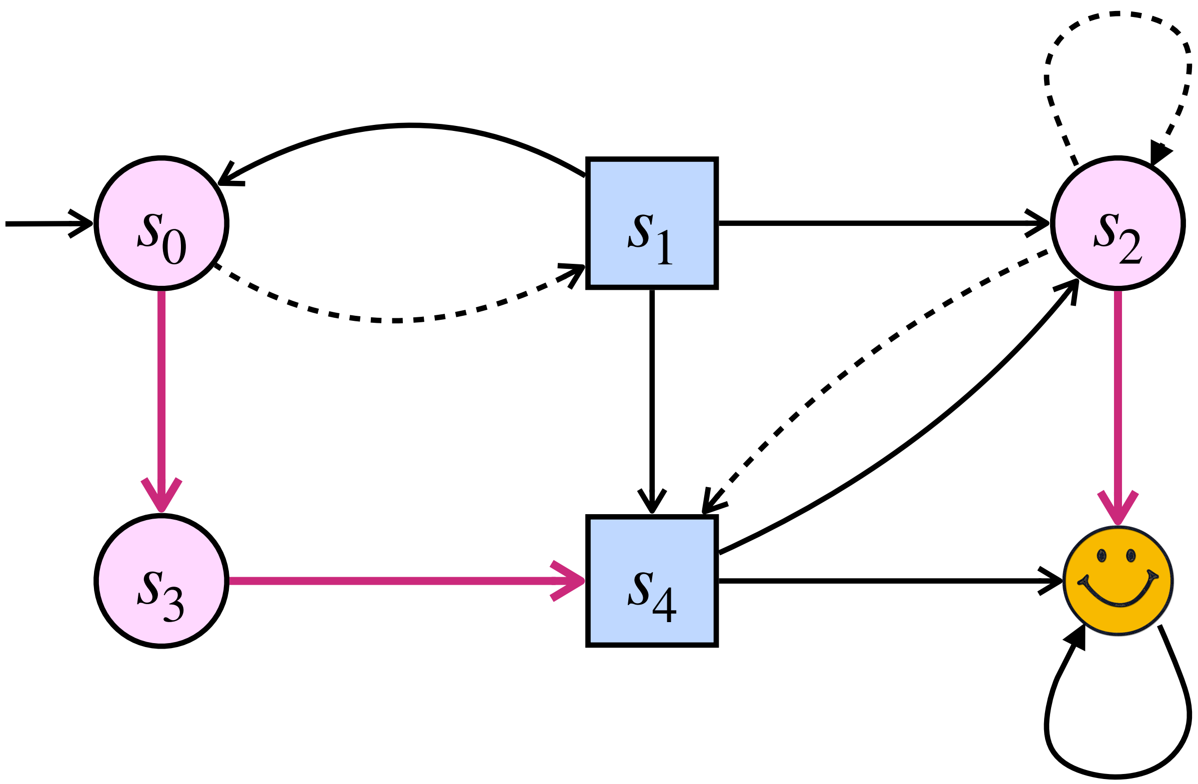
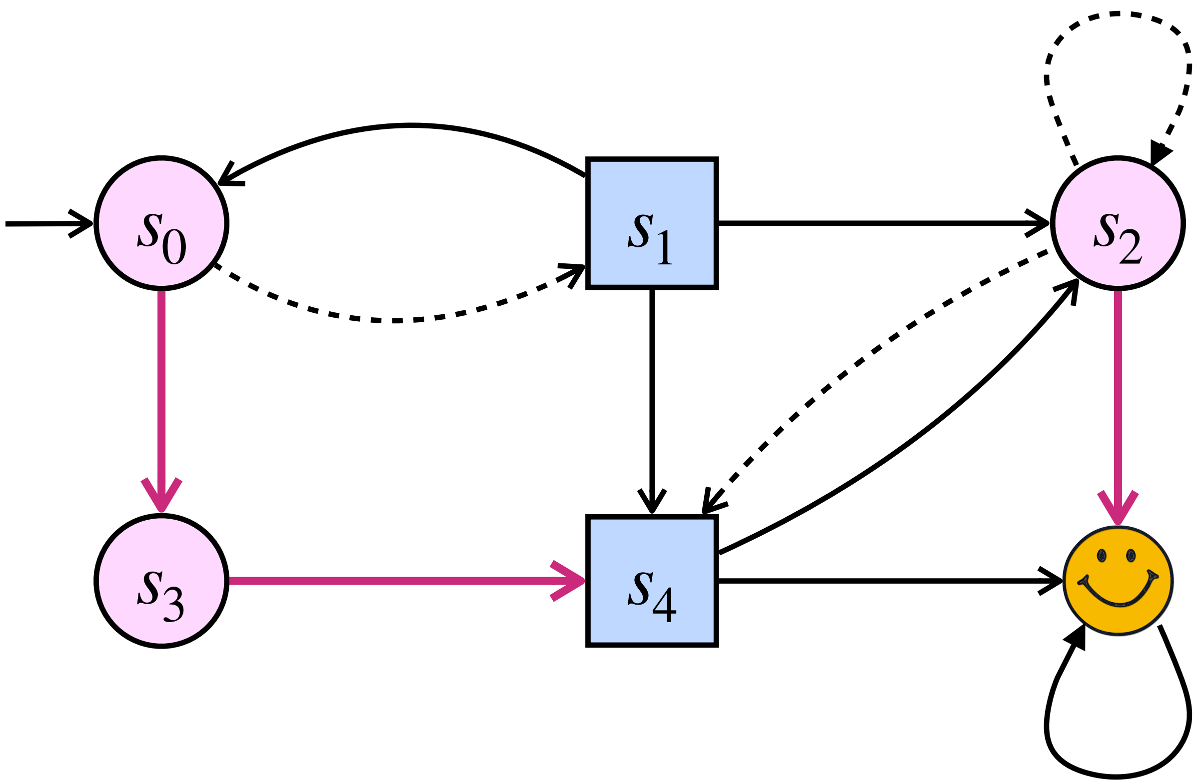
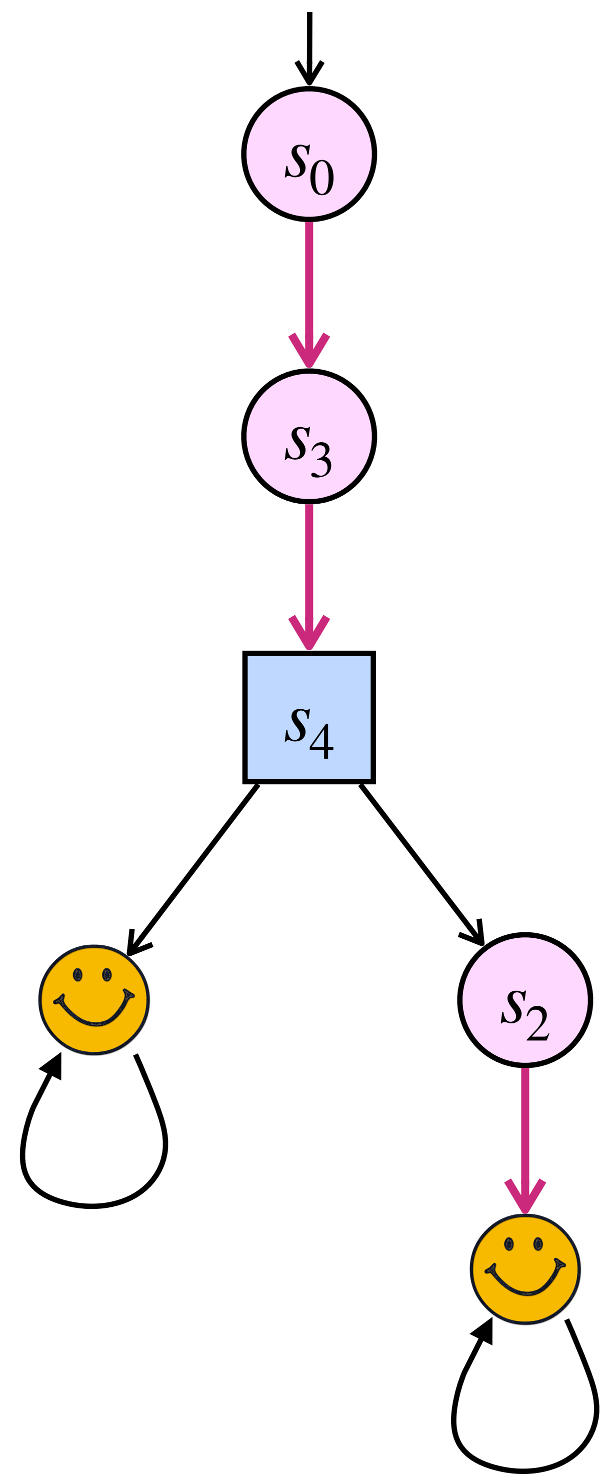


- Strategy σ

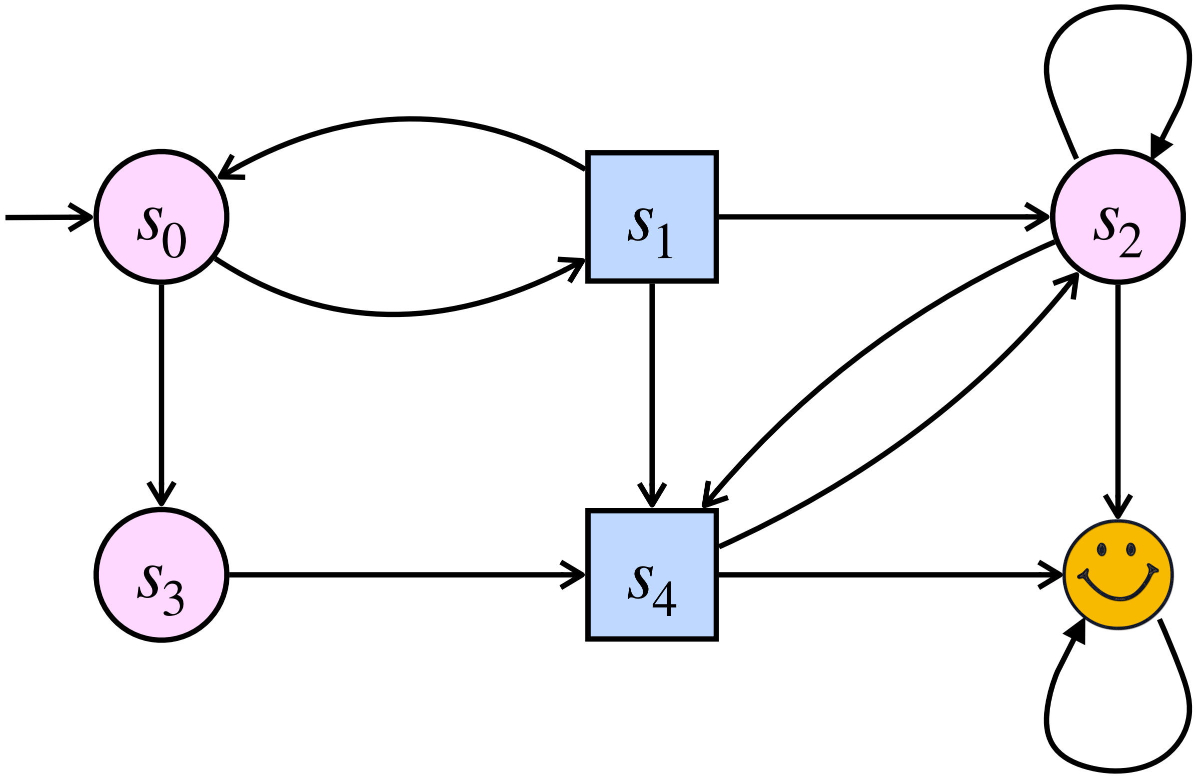
Outcomes of a strategy





- Strategy σ
- $\mathrm{Out}(\sigma)$ has two plays, which are both winning

Outcomes of a strategy



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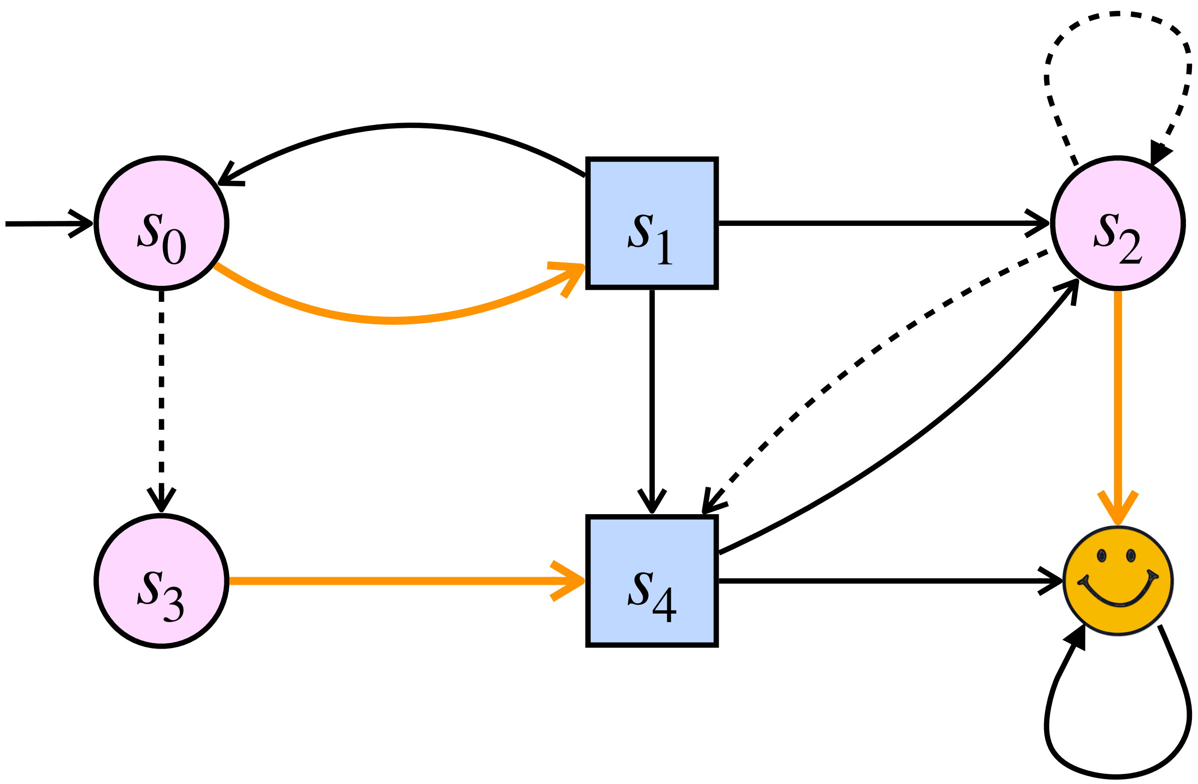
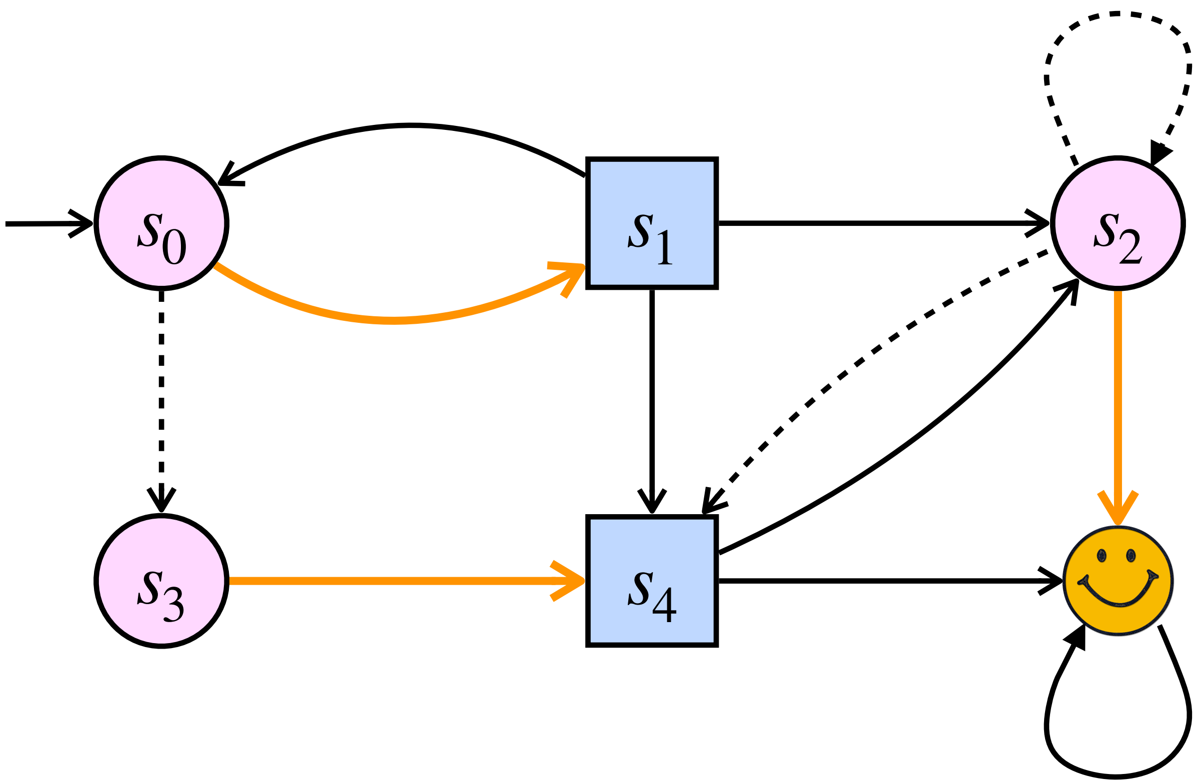
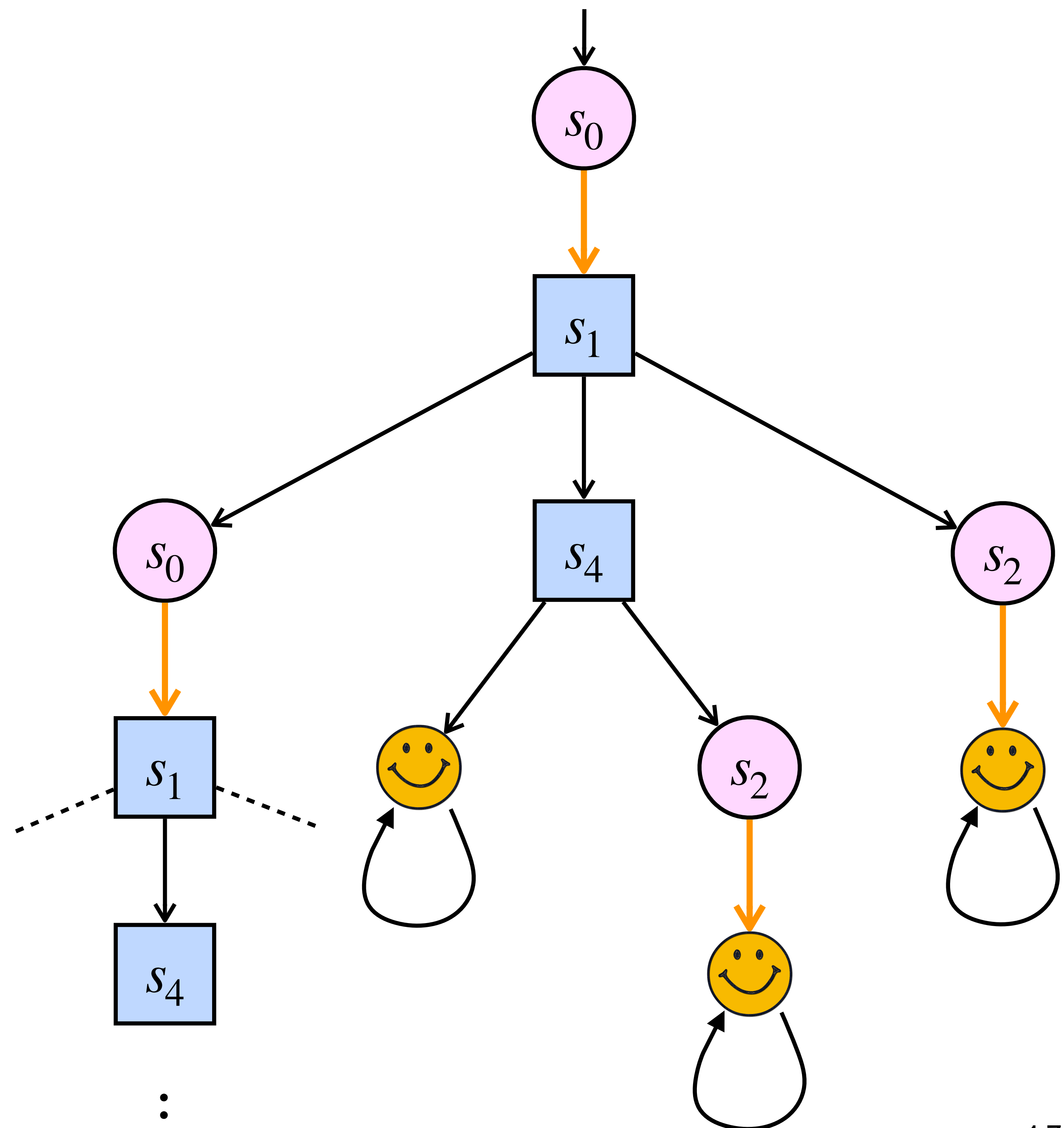


- Strategy σ

Outcomes of a strategy





- Strategy σ
- $\mathrm{Out}(\sigma)$ has infinitely many plays, some of them are not winning

What does it mean to win a game?

- Play $\rho = s_0 s_1 s_2 \ldots$ is compatible with σ_i whenever $s_j \in S_i$ implies $(s_j, s_{j+1}) = \sigma_i\left(s_0 s_1 \ldots s_j\right)$. We write Out($\sigma_i$).

- σ_i is **winning** if all plays compatible with σ_i belong to W_i
  σ_i is **optimal** if it is winning or if the initial state is losing

[Mar75] Martin. Borel determinacy (Annals of Mathematics).

What does it mean to win a game?

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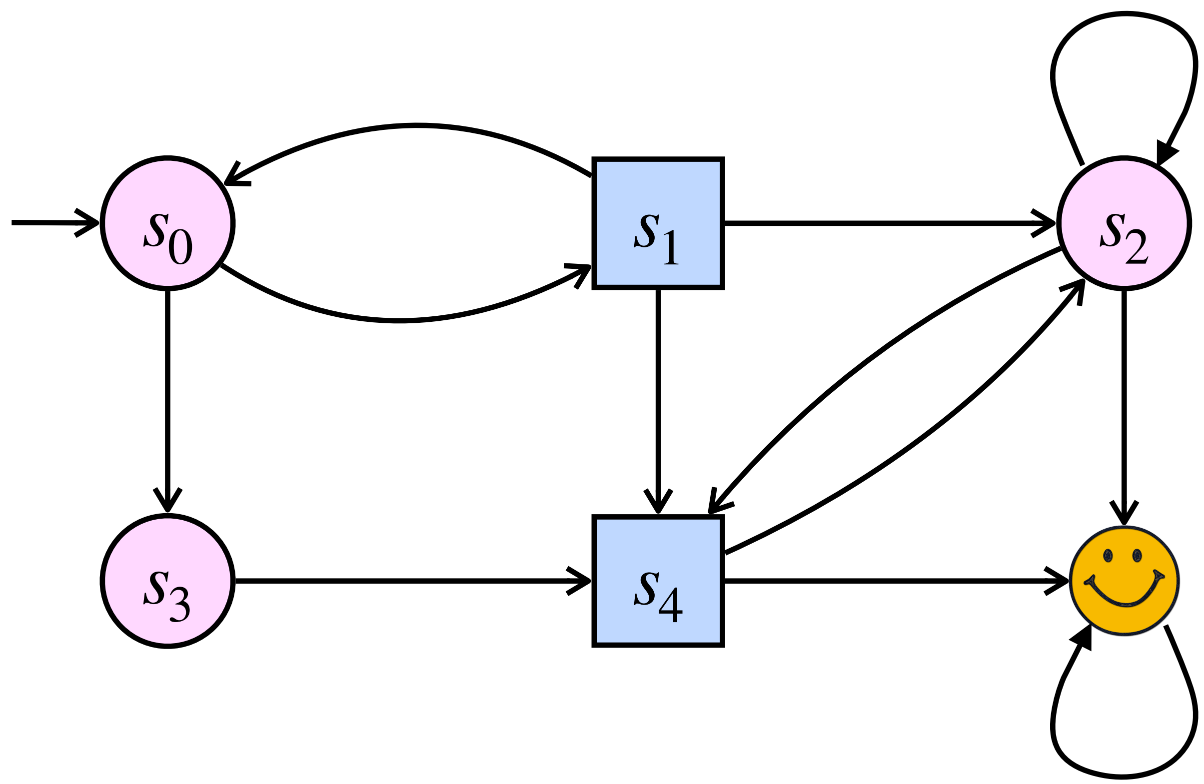
Martin's determinacy theorem

Turn-based zero-sum games are determined for Borel winning objectives: in every game, either P_1 or P_2 has a winning strategy.

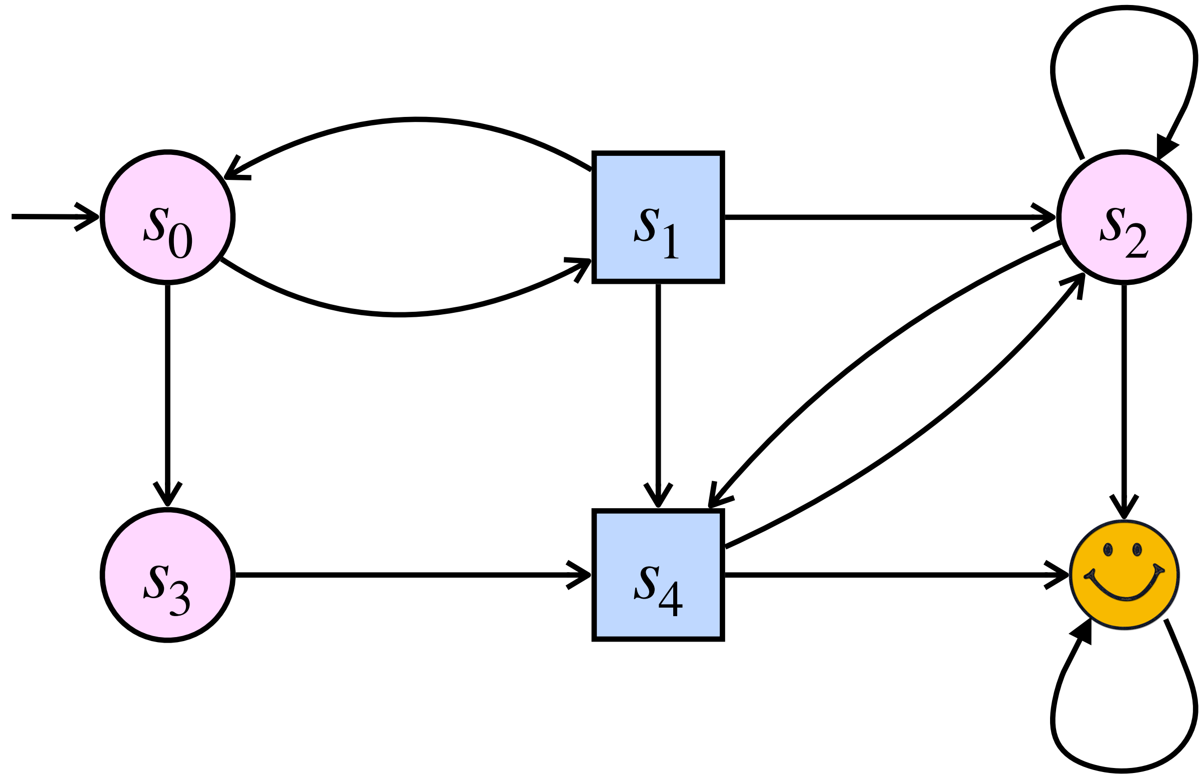
[Mar75] Martin. Borel determinacy (Annals of Mathematics).

Relevant questions



φ = Reach(☺)

Relevant questions



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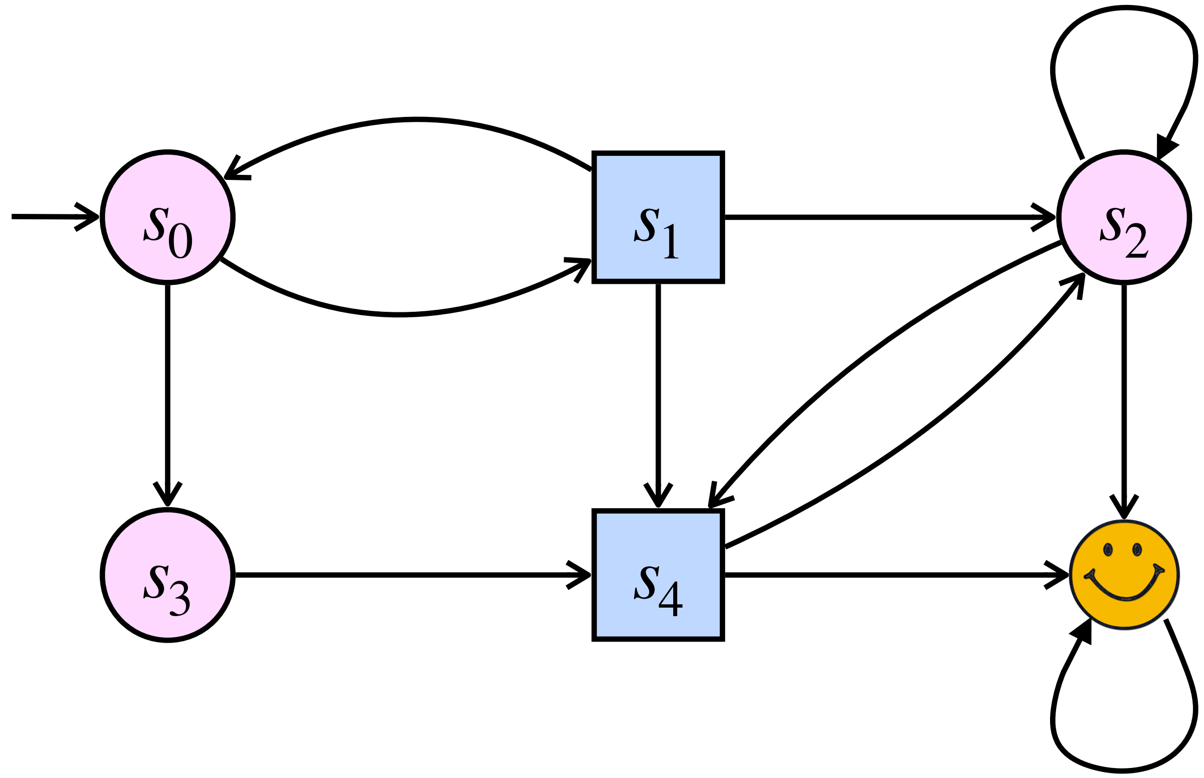

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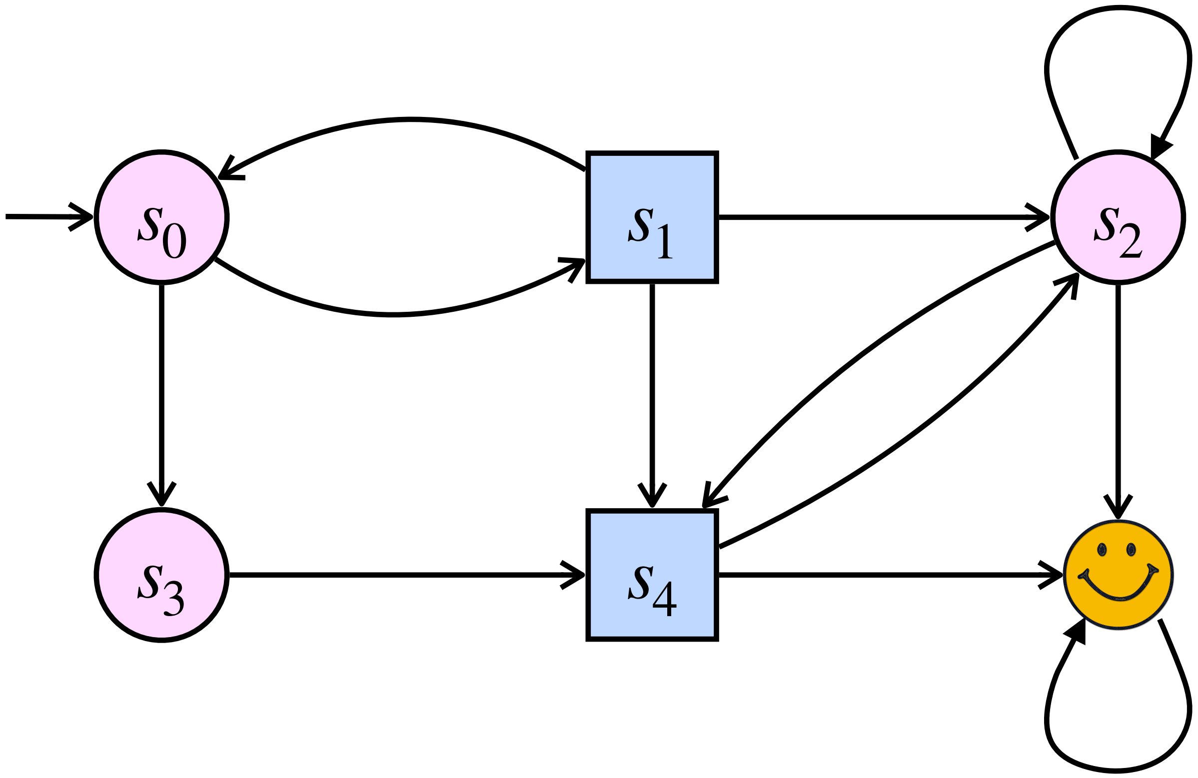

φ = Reach(☺)

- Can P_1 win the game, i.e. does P_1 have a winning strategy?
- Is there an effective (efficient) way of winning?
- How complex is it to win?

Example: the Nim game

- Players alternate
- Each player can take one or two sticks
- The player who takes the last one wins
- P_1 starts

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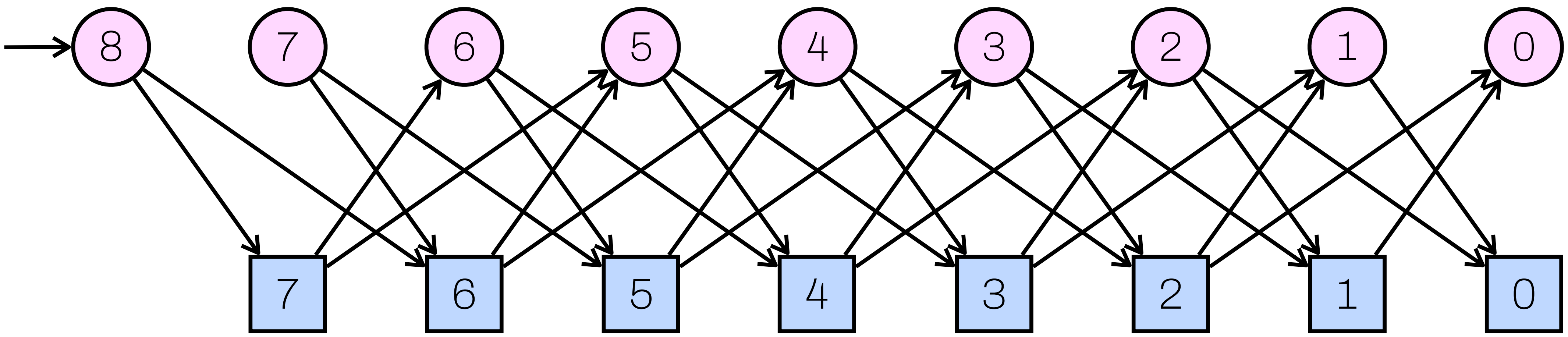

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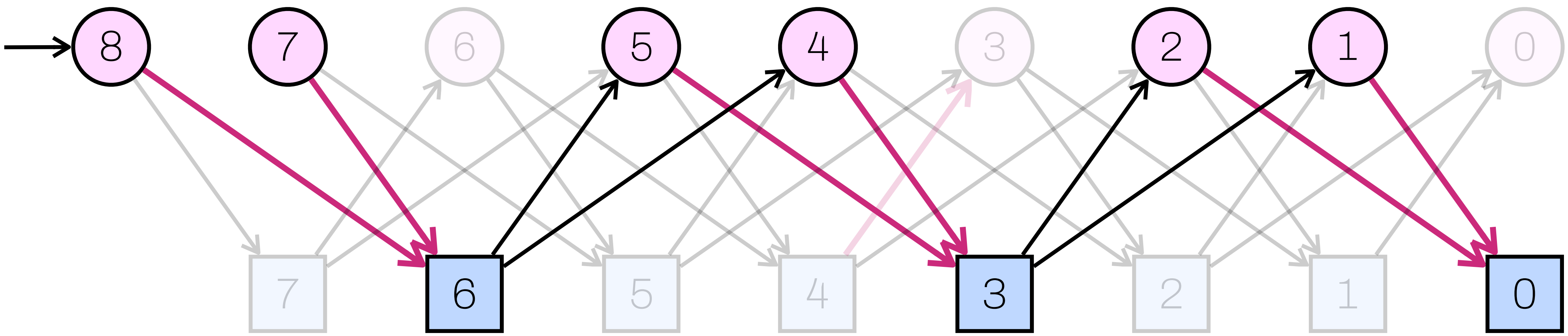

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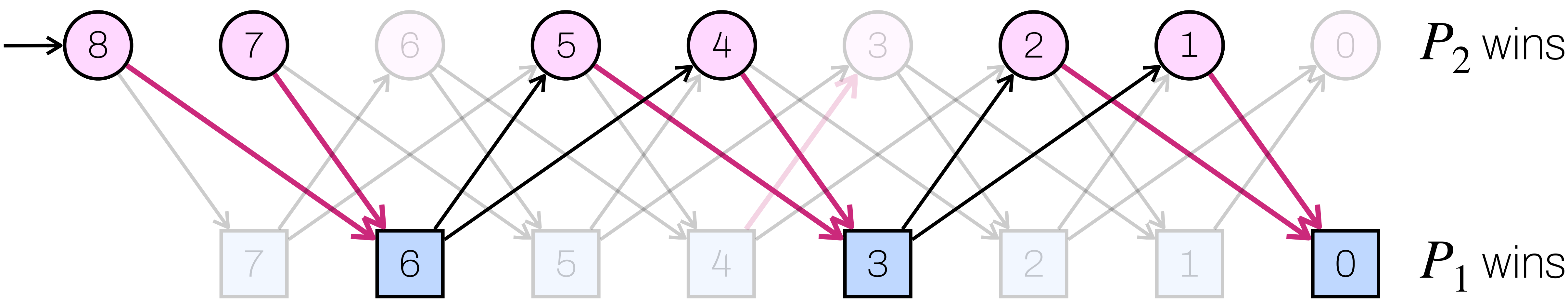

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- from all ◯ $\equiv 1$ or $2 \mod 3$
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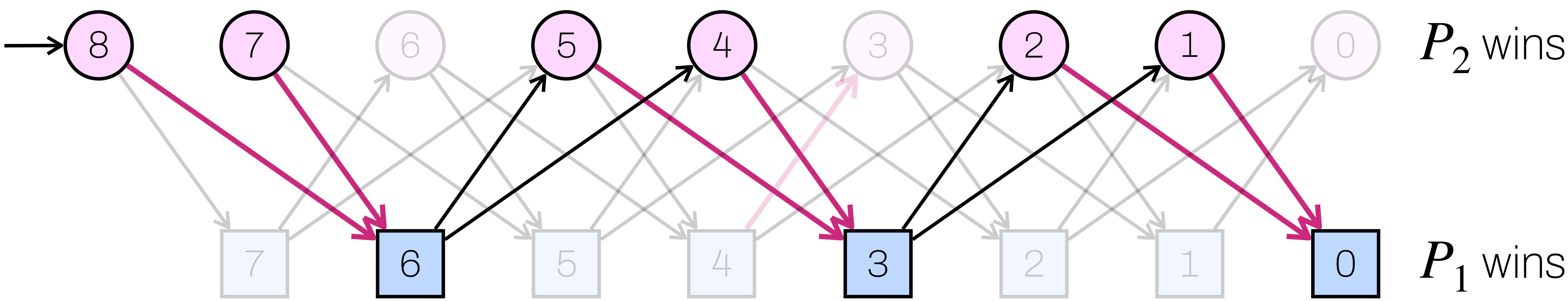


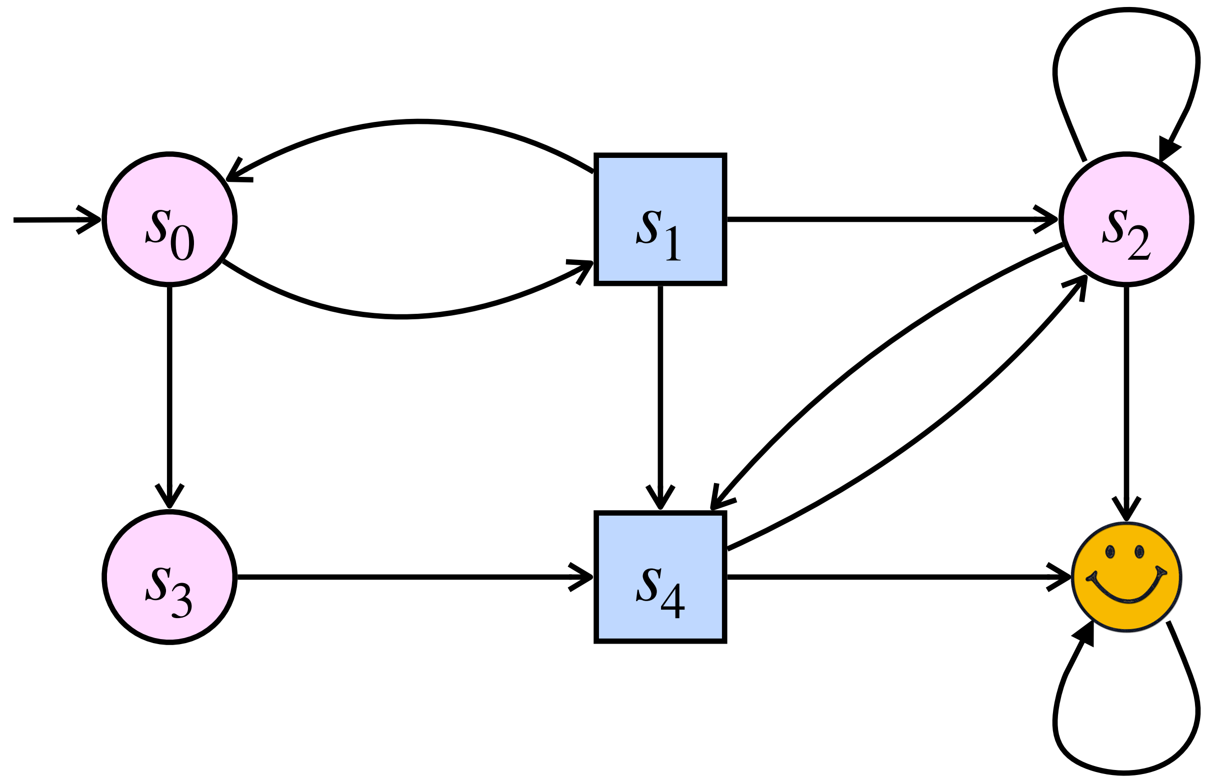
P_1 wins

- from all ○ $\equiv 1$ or $2 \mod 3$
- from all □ $\equiv 0 \mod 3$

P_2 wins

- from all ○ $\equiv 0 \mod 3$
- from all □ $\equiv 1$ or $2 \mod 3$

Computation of winning states in the running example



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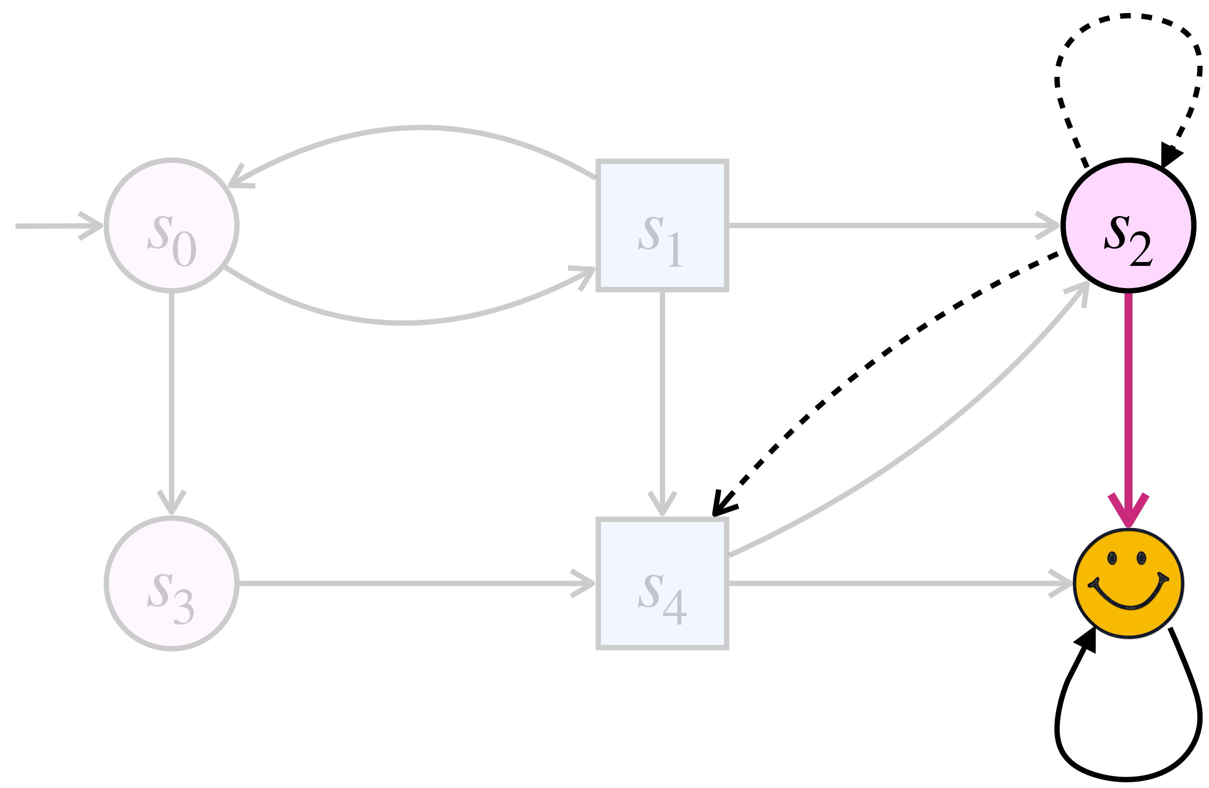

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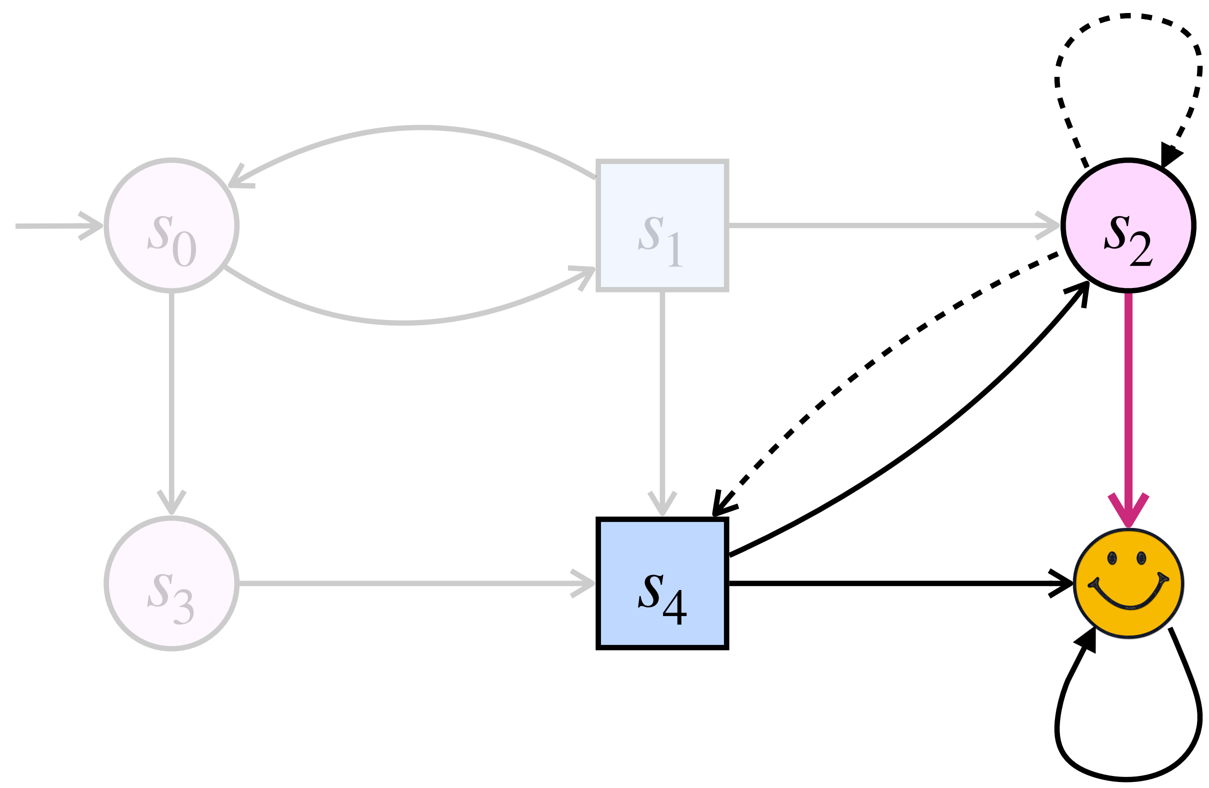

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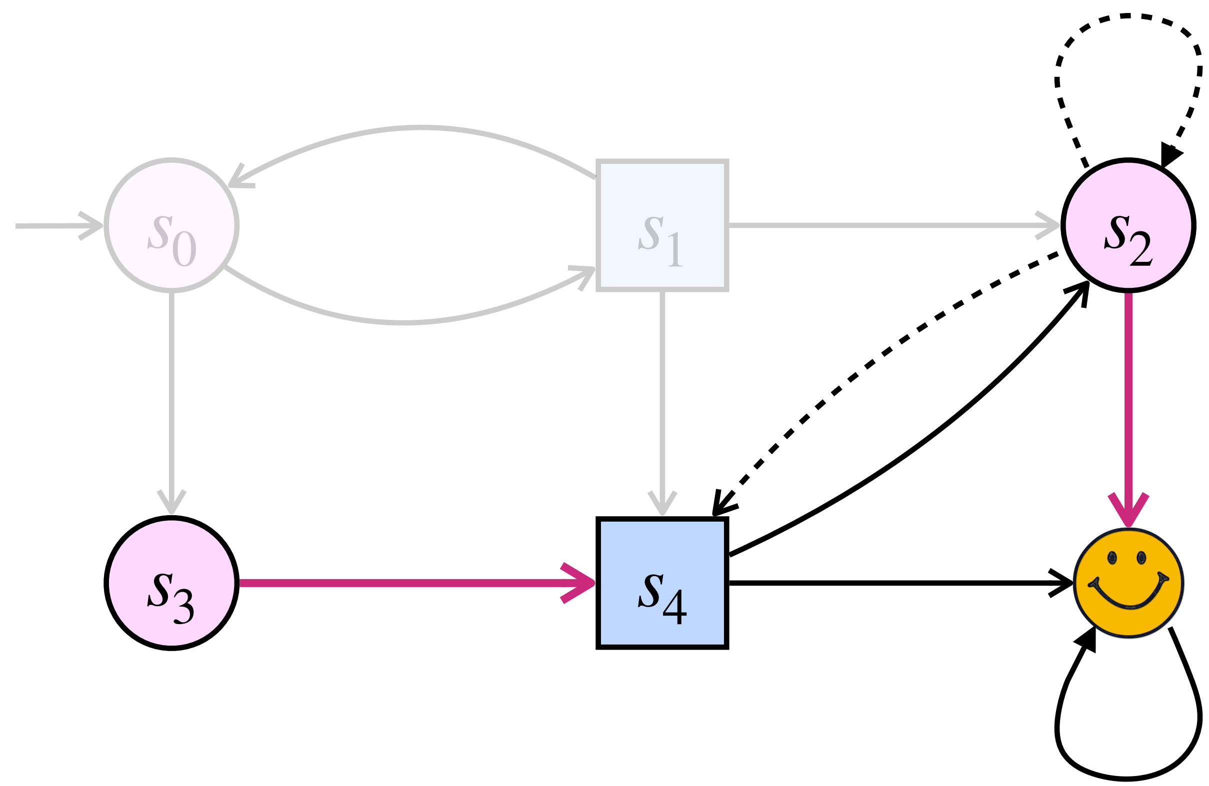

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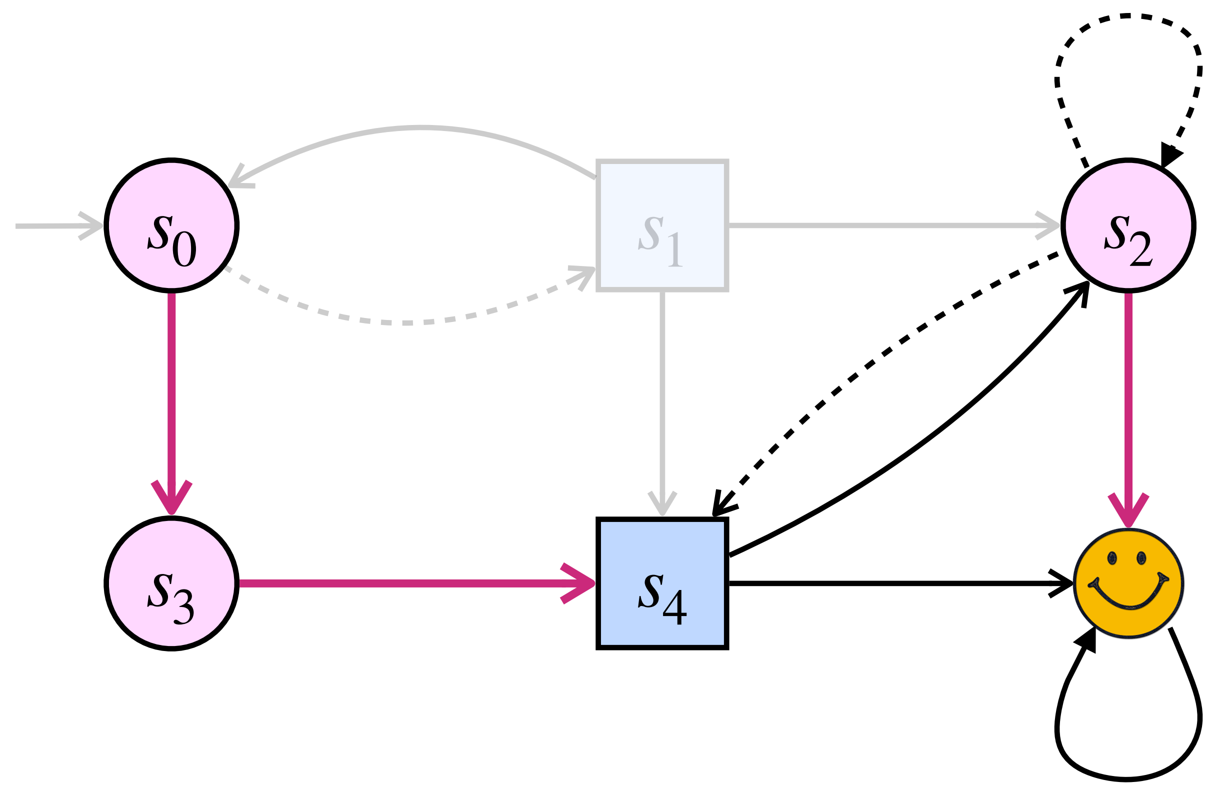

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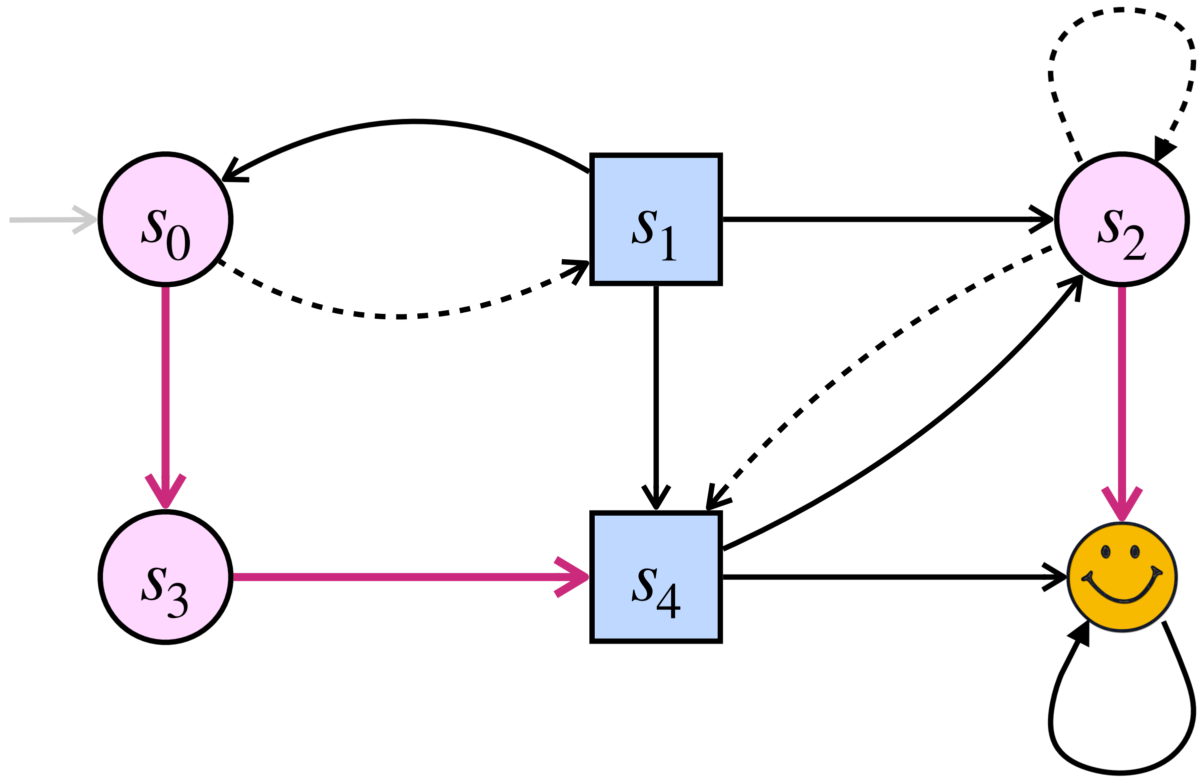

All states are winning for P_1

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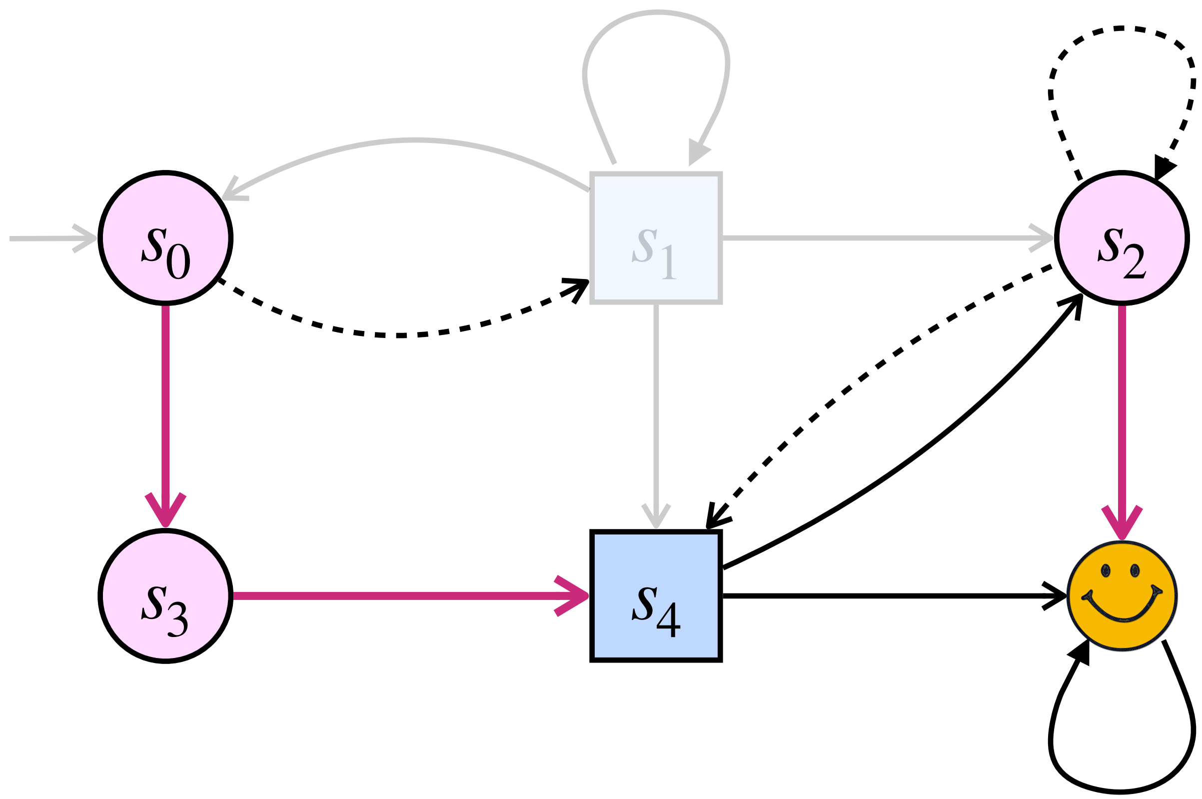


One state is not winning for P_1

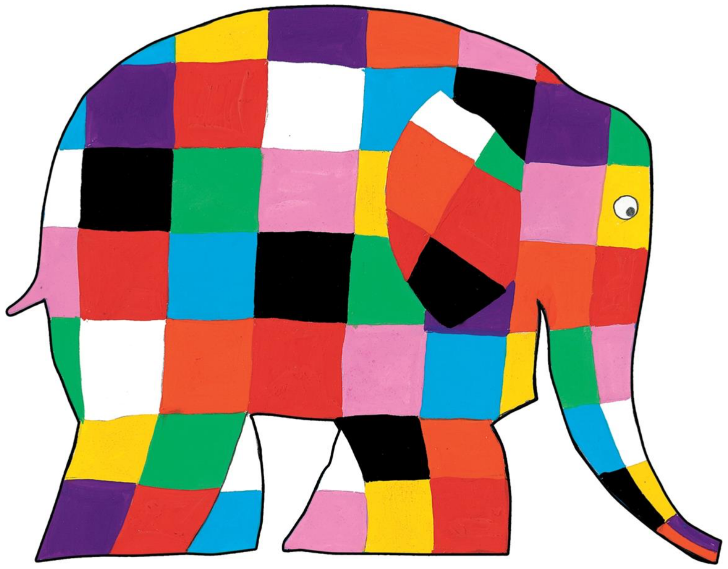
It is winning for P_2

What we do not consider

- Concurrent games
- Stochastic games and strategies
  - Values
  - Determinacy of Blackwell games
- Partial information

Families of strategies

Families of strategies



General strategies

$$\sigma_i : S^*S_i \rightarrow E$$

- May use any information of the past execution
- Information used is therefore potentially infinite
- Not adequate if one targets implementation

On the simplest side: positional strategies

From $\sigma_i : S^*S_i \to E$ to $\sigma_i : S_i \to E$

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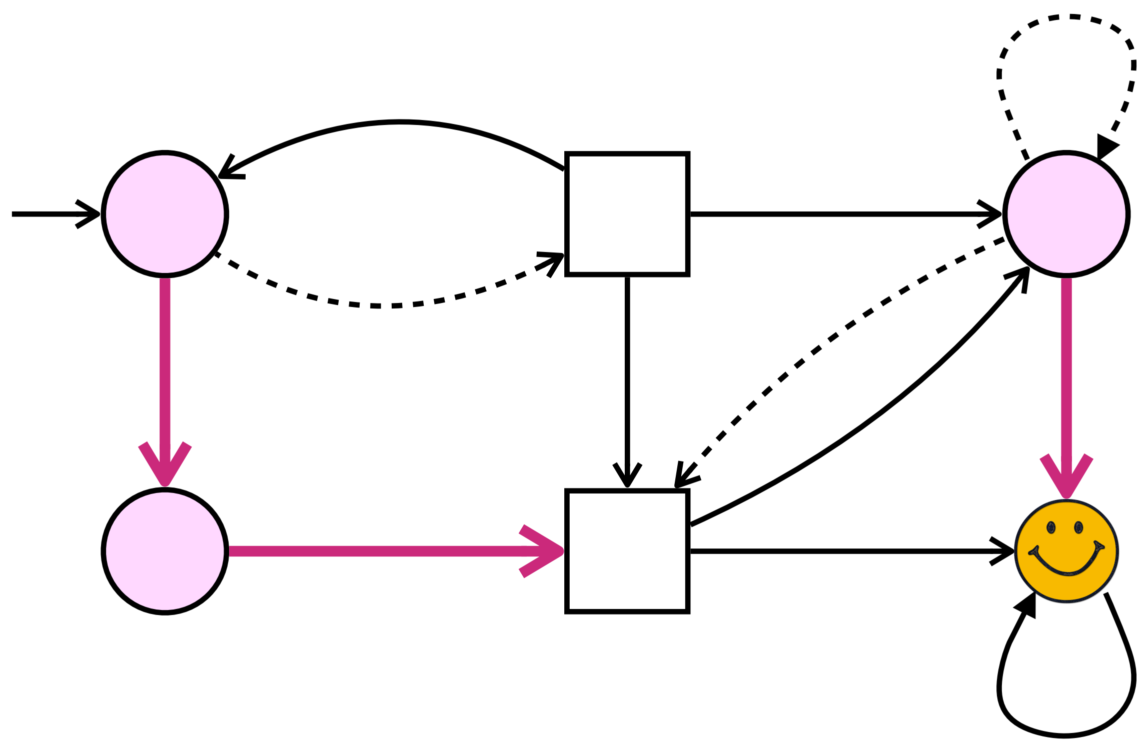
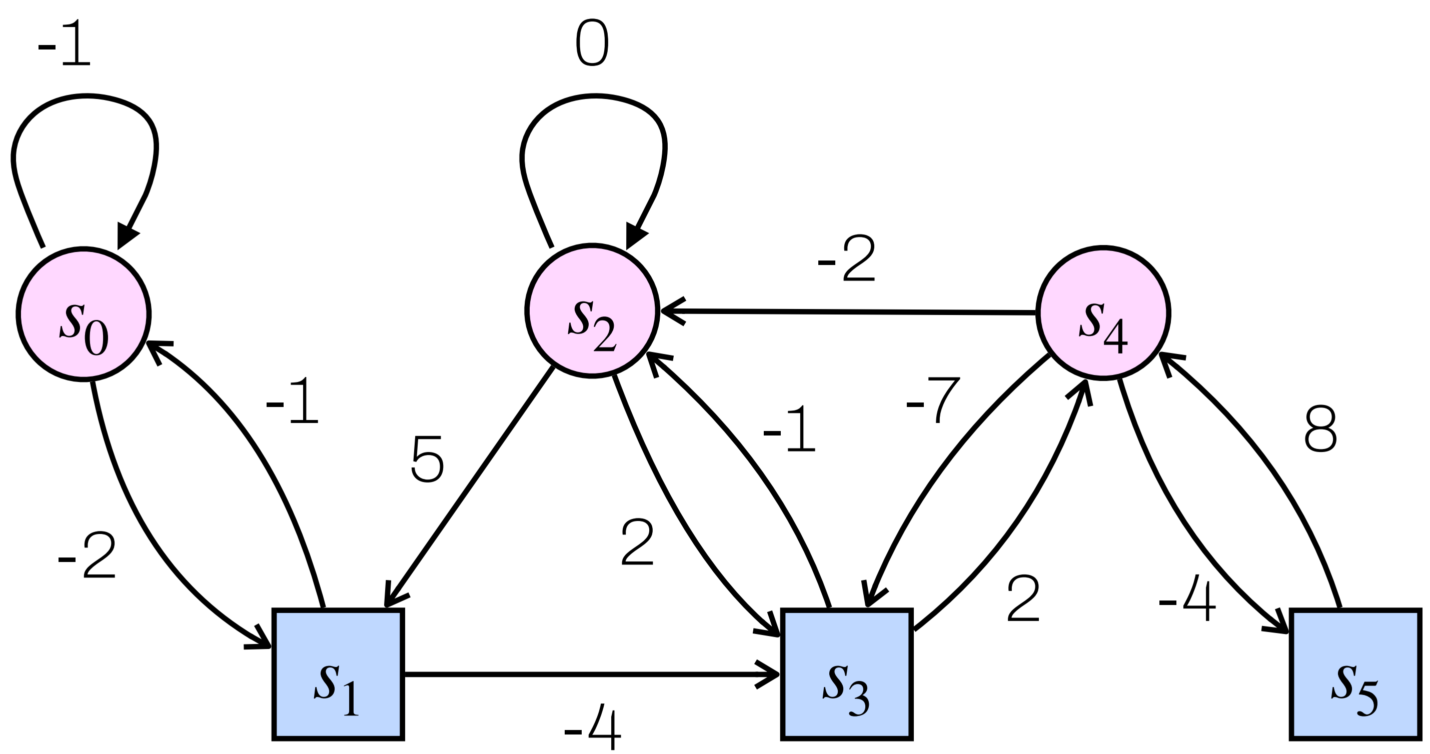


Example: mean-payoff

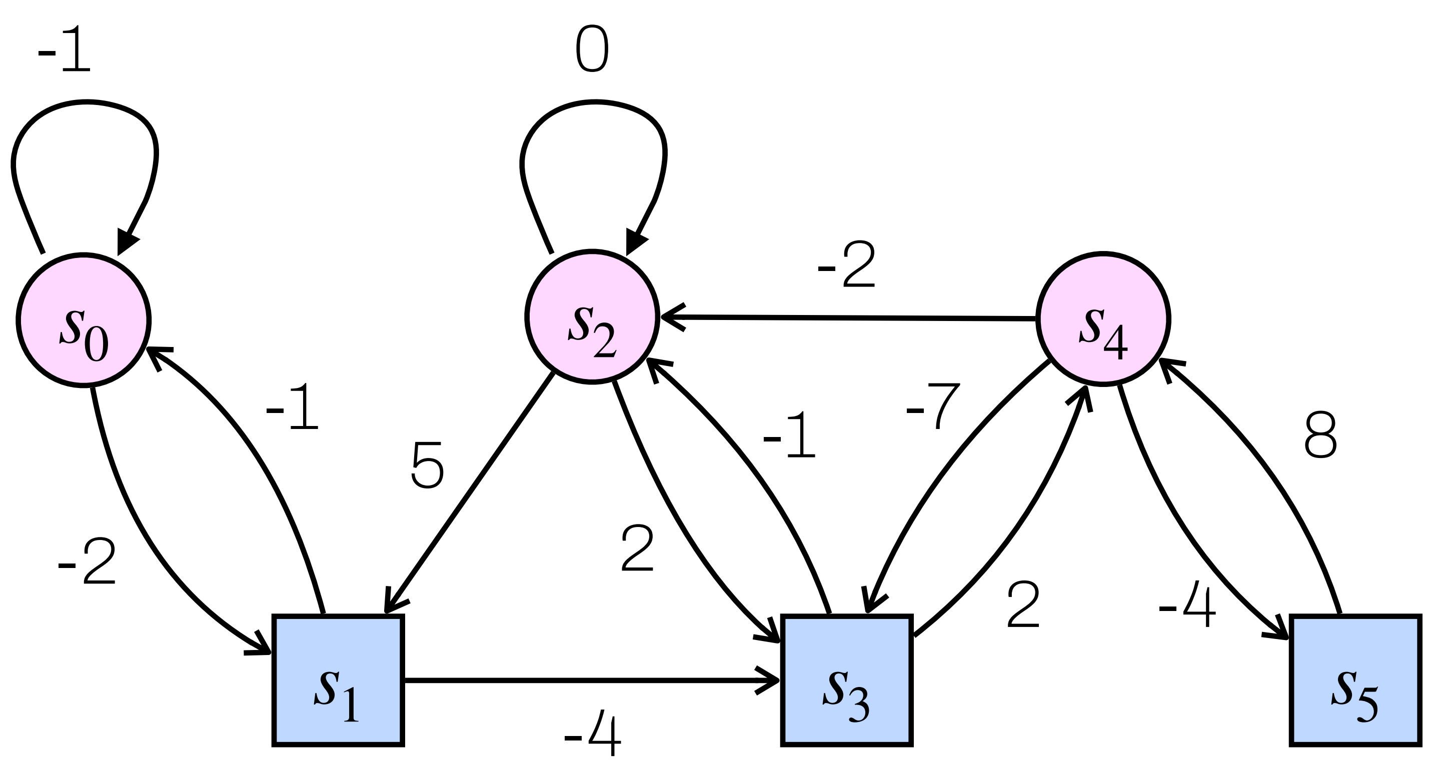


[Ohl21] Ohlmann. Monotonic graphs for Parity and Mean-Payoff games (PhD thesis).

Example: mean-payoff

- P_1 maximizes, P_2 minimizes

$$\overline{\mathsf{MP}} = \limsup_n \frac{\sum_{i \neq n} c_i}{n}$$



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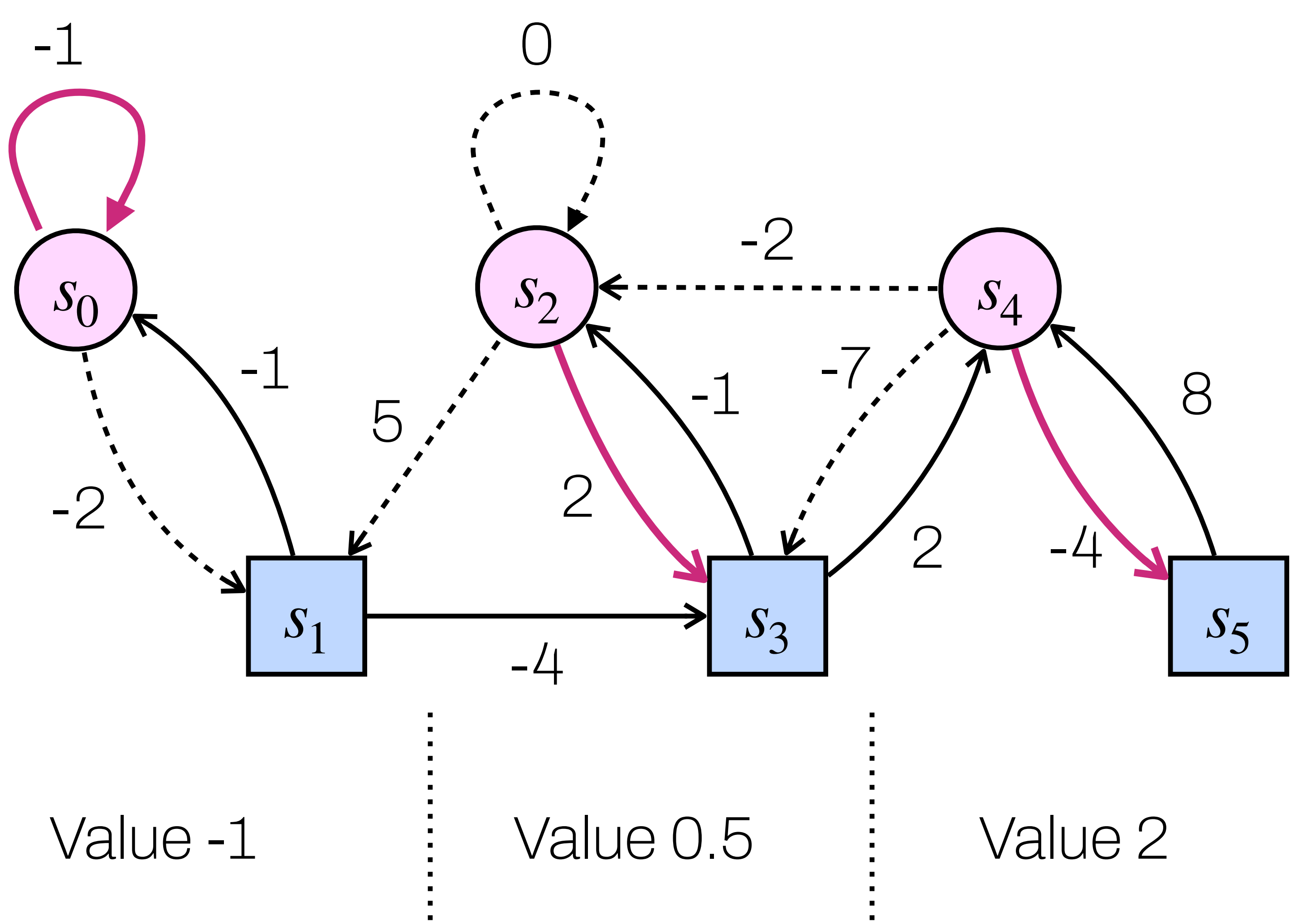

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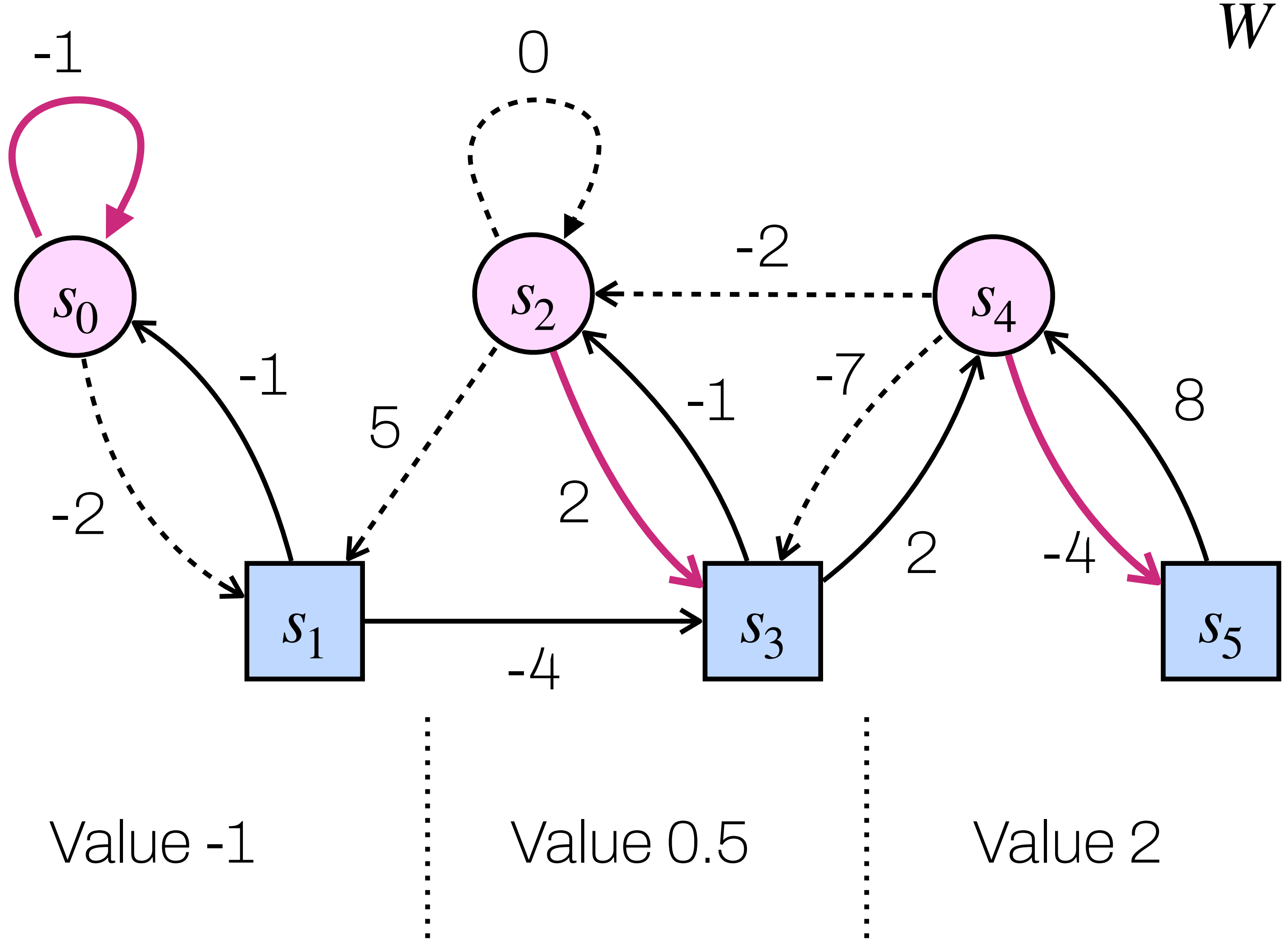

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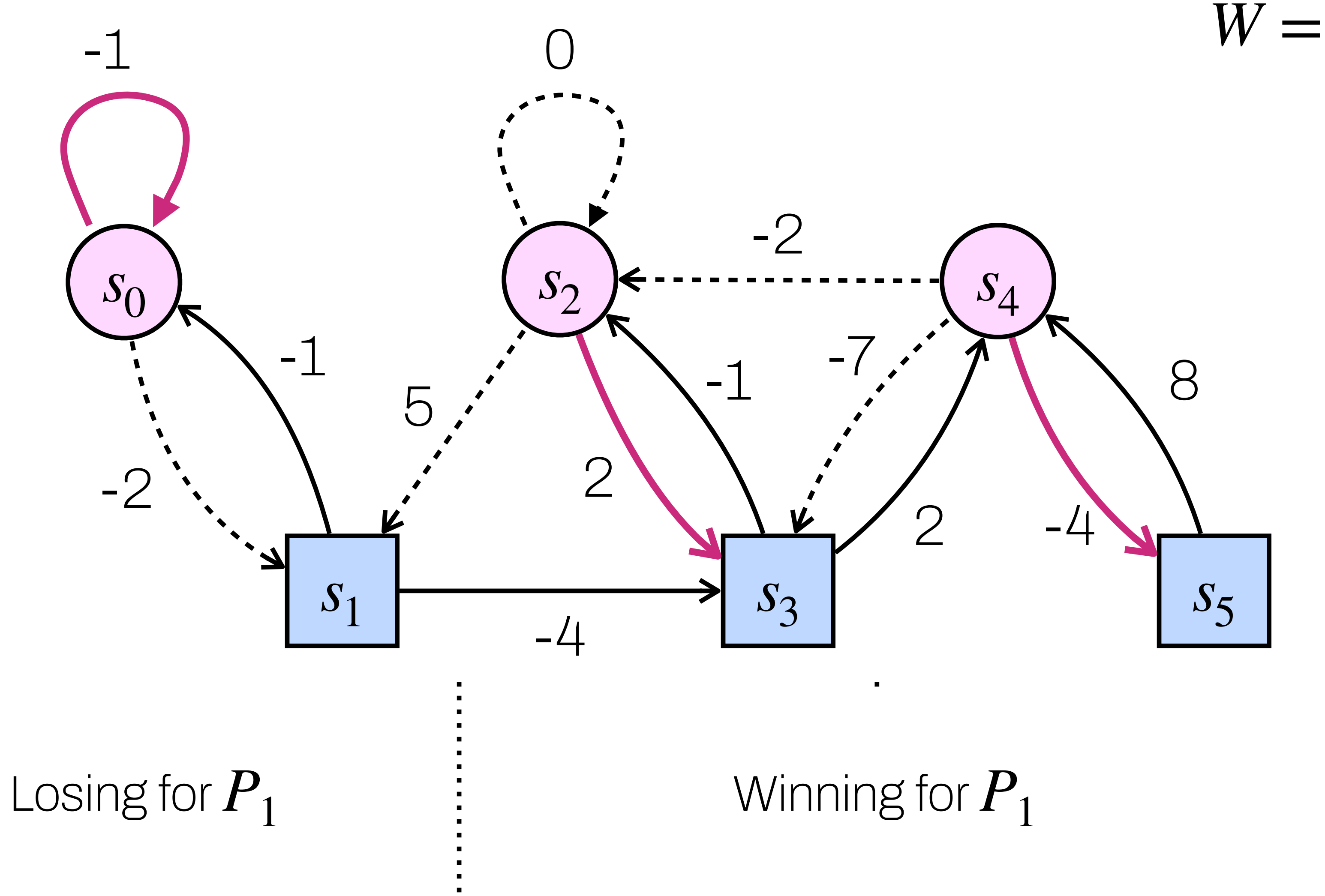

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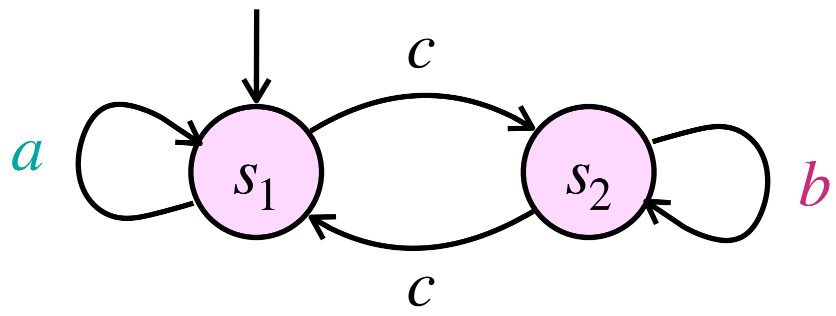
Do we need more?

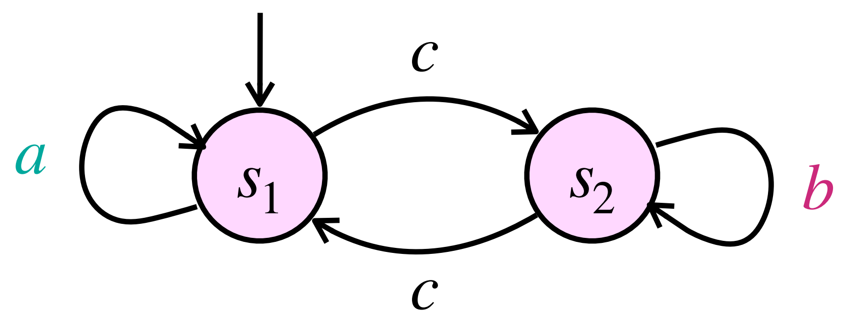
Examples



« See infinitely often both a and b »

Büchi$(a) \wedge$ Büchi(b)

Examples



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Winning strategy

- At each visit to s_1, loop once in s_1 and then go to s_2
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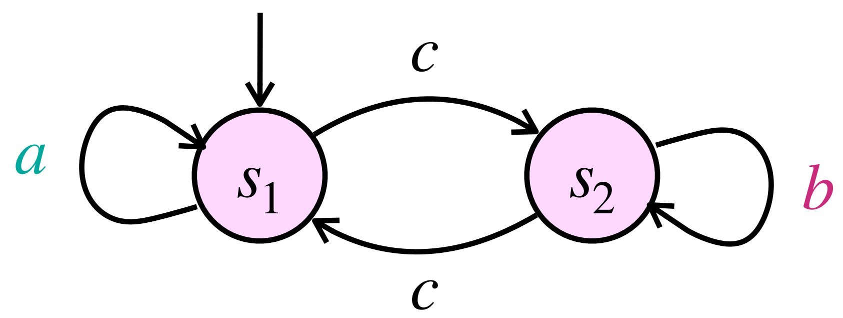
- Generates the sequence $(acbc)^\omega$

Examples



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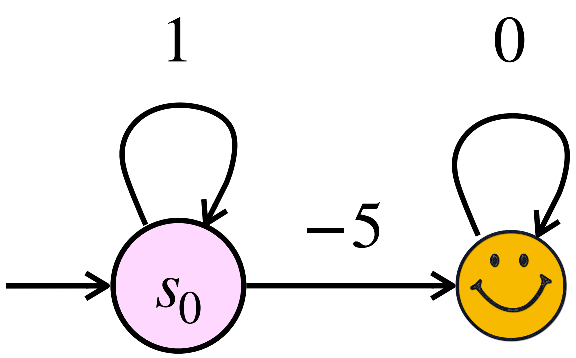
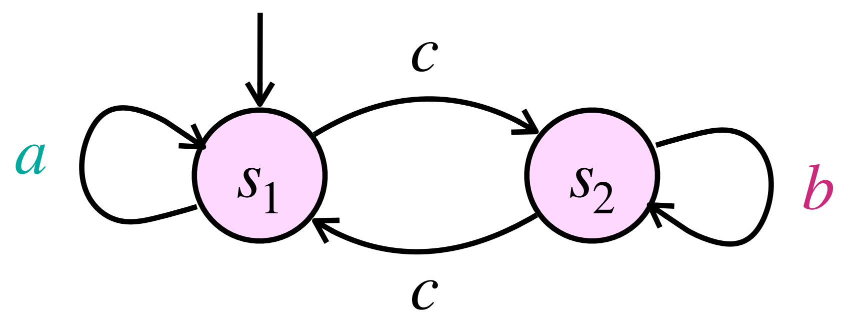


« Reach the target with energy level 0 »

$\mathbf{FG}\ (\mathsf{EL} = 0)$

Examples



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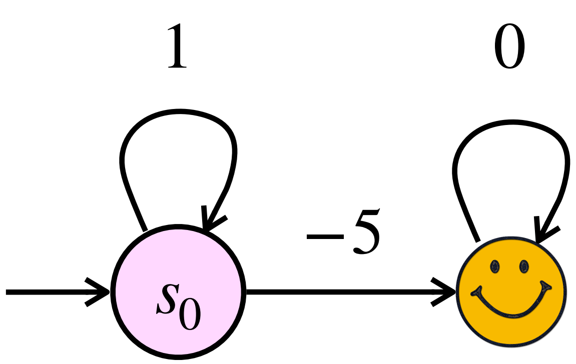

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- Loop five times in s_0
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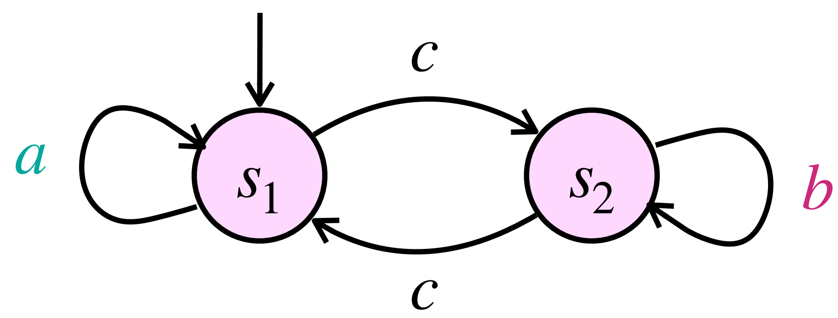
Examples



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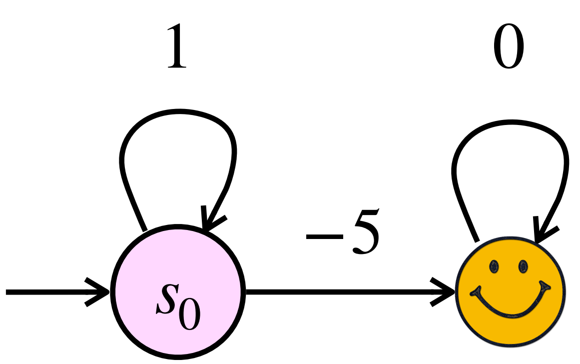

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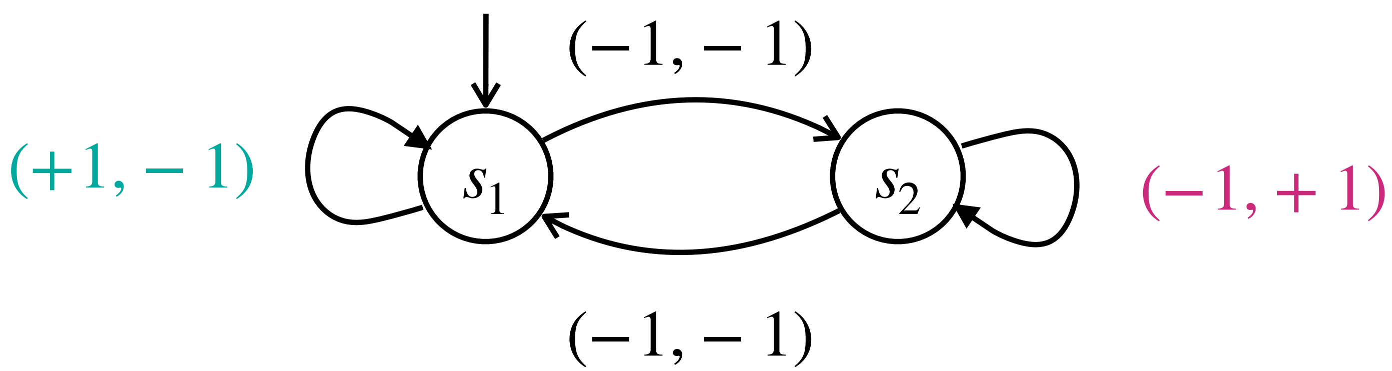
These two strategies require only **finite** memory

Example: multi-dimensional mean-payoff



« Have a (limsup) mean-payoff ≥ 0 on both dimensions »

So-called *multi-dimensional mean-payoff*

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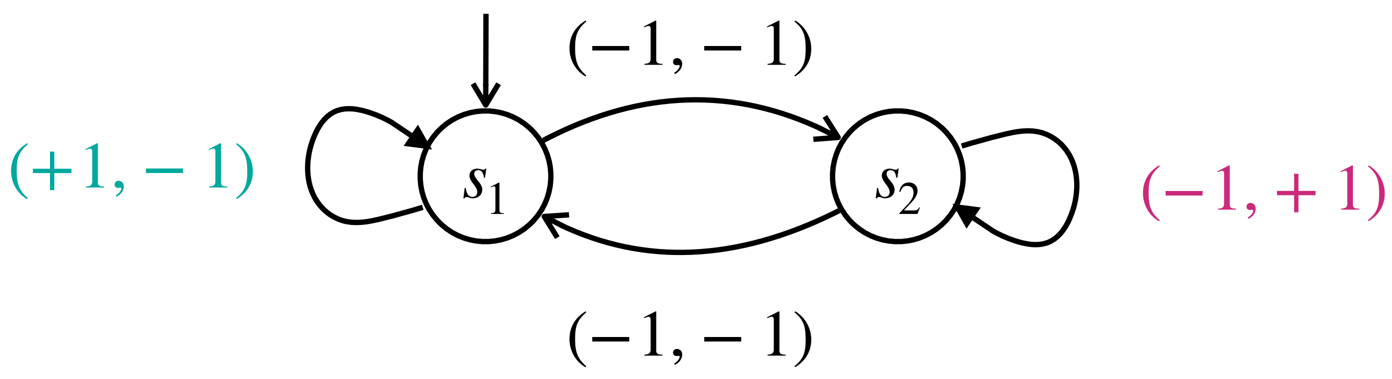

« Have a (limsup) mean-payoff ≥ 0 on both dimensions »

So-called *multi-dimensional mean-payoff*

Winning strategy

- After k-th switch between s_1 and s_2, loop $2k - 1$ times and then switch back
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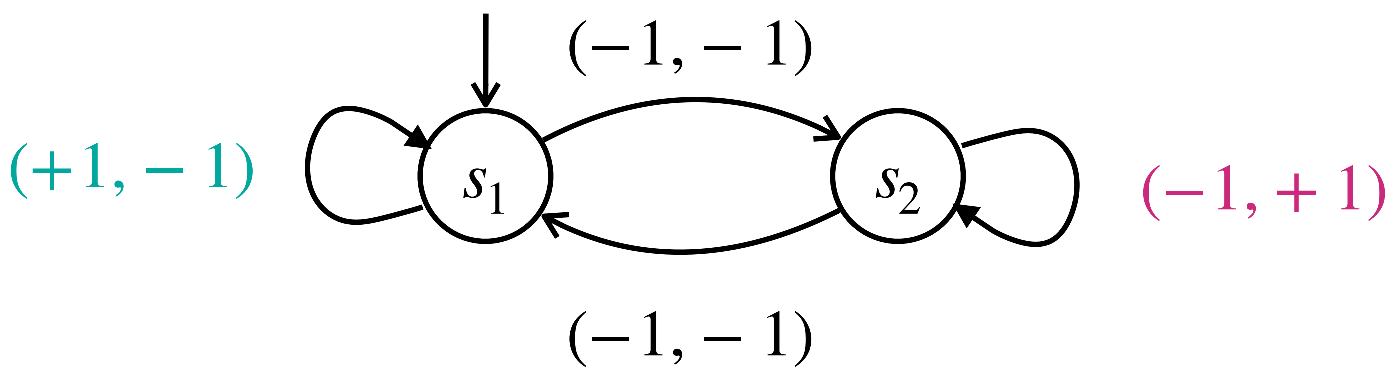

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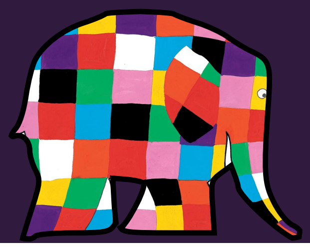
$(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)(+1,-1)\,(-1,-1)$...

This strategy requires **infinite** memory, and this is unavoidable

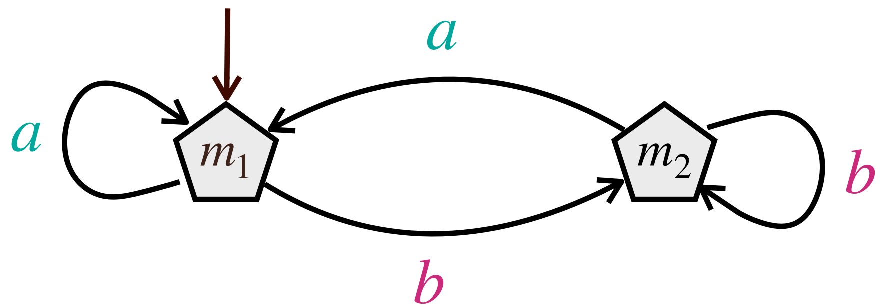
We focus on finite memory!



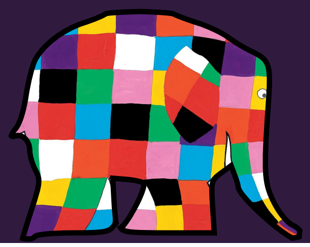
Chromatic* memory

Memory skeleton

$\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ with $m_{\text{init}} \in M$ and $\alpha_{\text{upd}} : M \times C \to M$



* Terminology by Kopczyński



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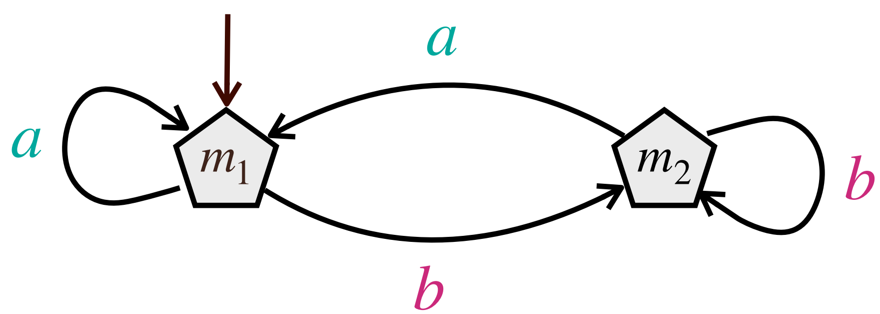

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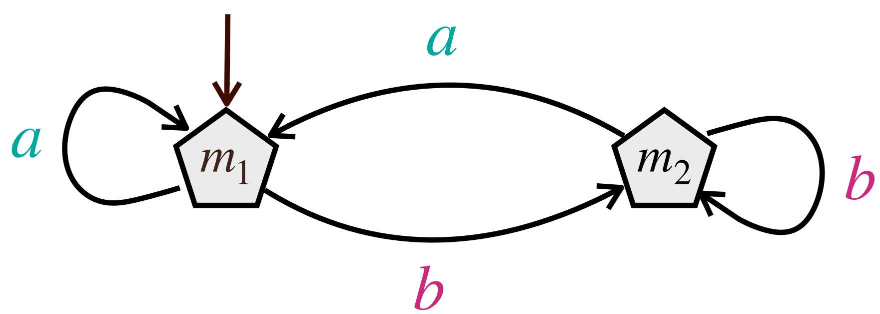

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Strategy with memory $\mathcal{M}$

Additional next-move function $\alpha_{\text{next}} : M \times S_i \to E$

$(\mathcal{M}, \alpha_{\text{next}})$ defines a strategy!

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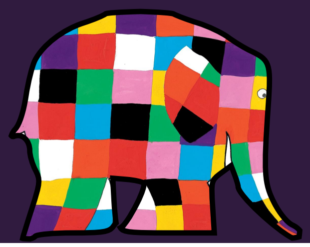

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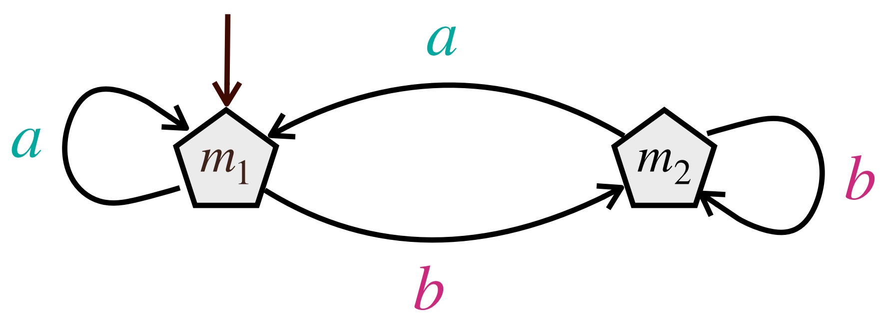

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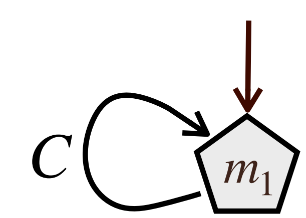

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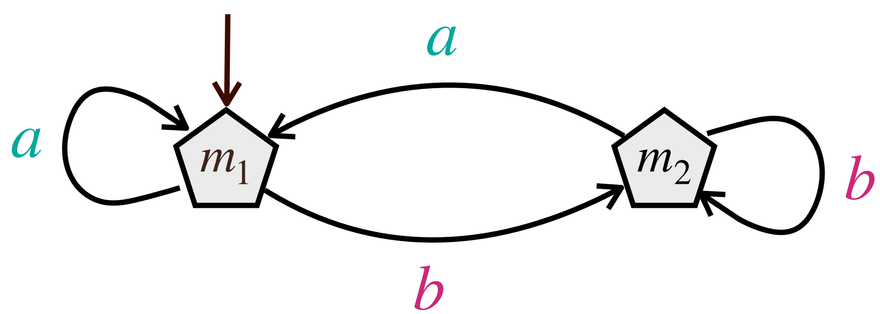
Chaotic* memory



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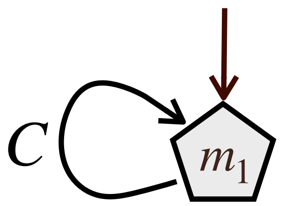
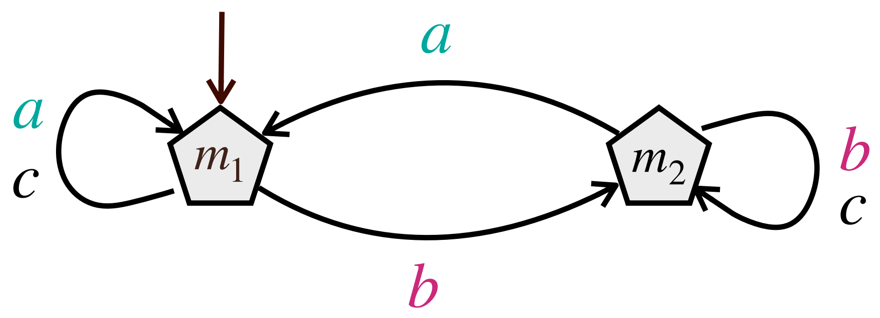


* Terminology by Kopczyński

Example of chromatic memory

$\mathscr{M}$



This skeleton is sufficient for winning

$W =$ Büchi$(a) \wedge$ Büchi(b) (in any arena)

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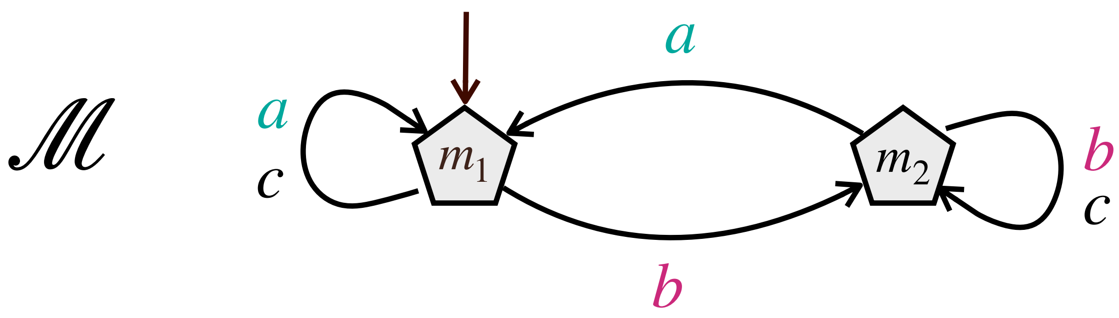

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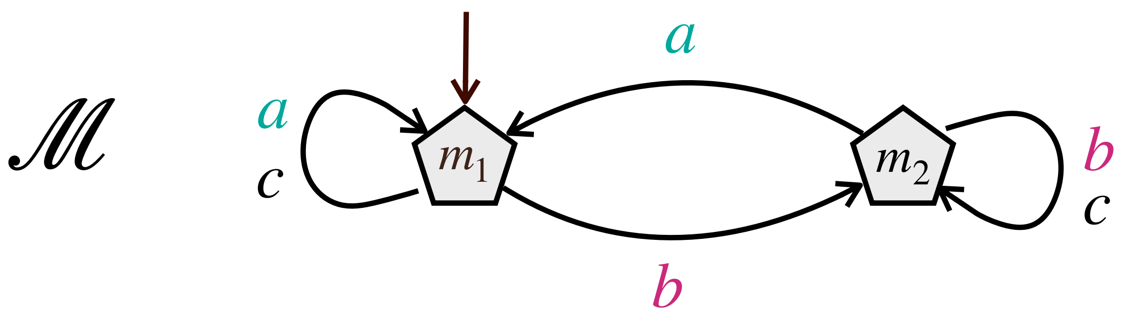

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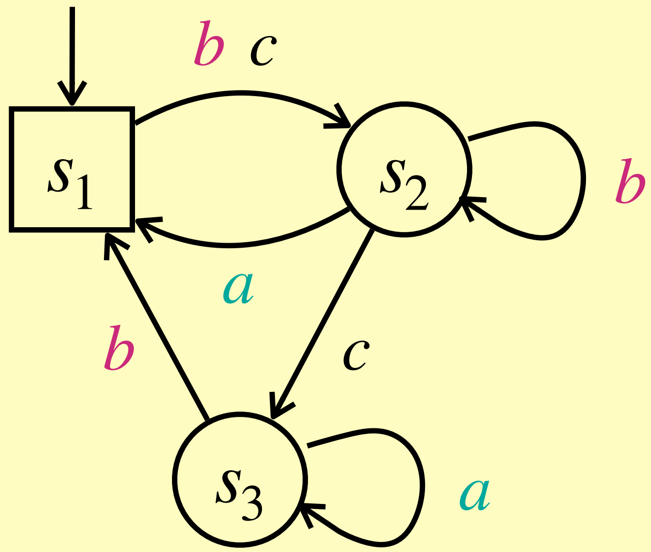

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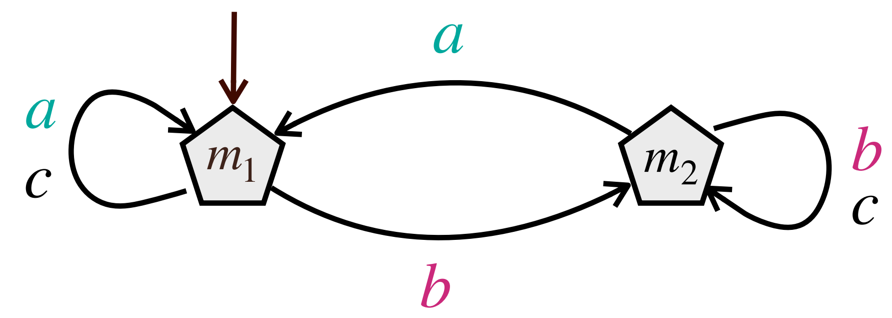

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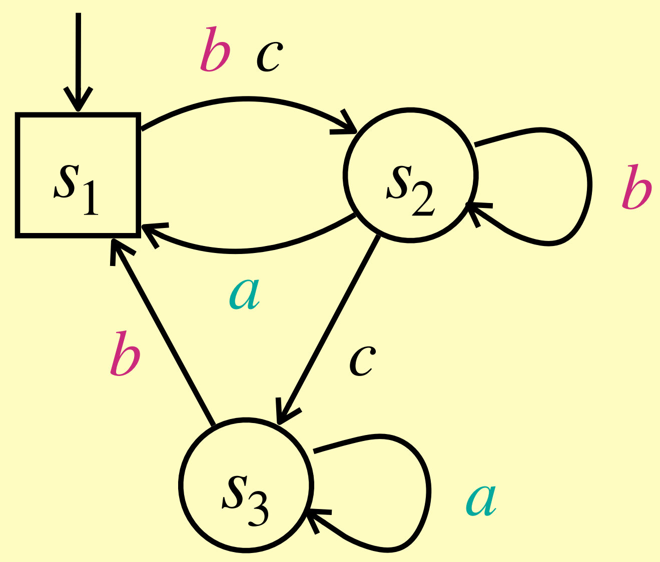
Example



$$\alpha_{\text{next}} : \quad M \times S_1 \to E$$
$$(m_1, s_2) \mapsto (s_2, b, s_2)$$
$$(m_2, s_2) \mapsto (s_2, a, s_1)$$
$$(m_\star, s_3) \mapsto (s_3, b, s_1)$$

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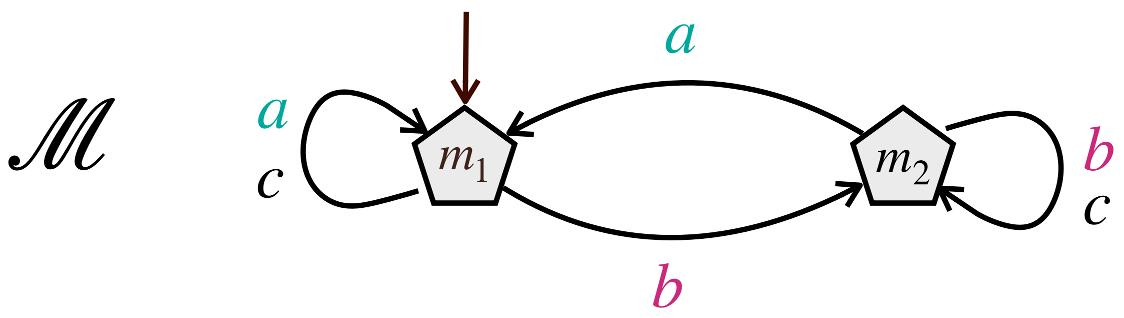

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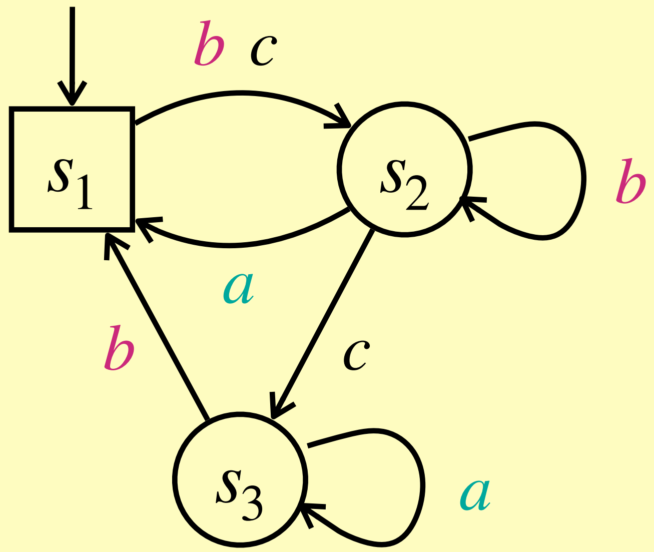

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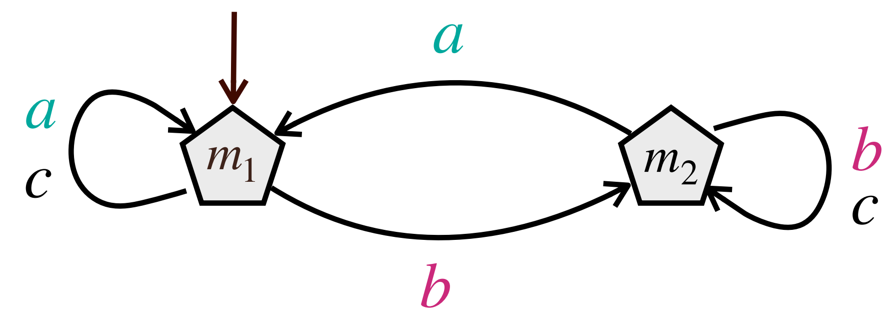

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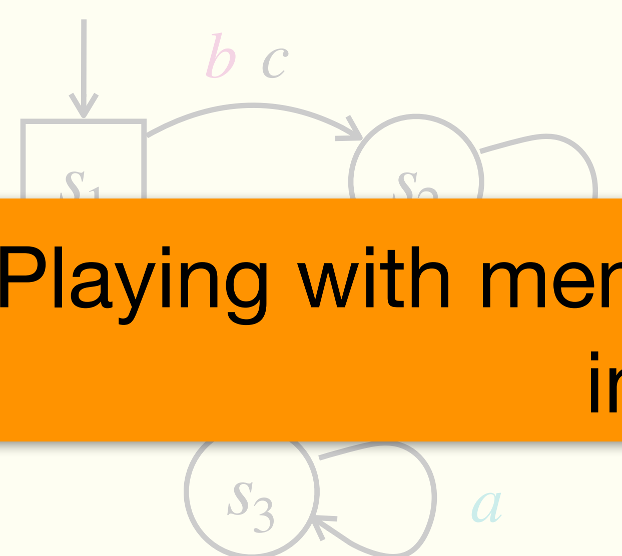

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Playing with memory $\mathscr{M}$ is like playing memoryless in the product arena

A zoology of notions

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- Let W be an objective and $i \in \{1,2\}$

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in finite arenas
in one-player arenas

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- W is half-positional = $\mathcal{M}_{\text{triv}}$ suffices to play optimally for P_1 for W

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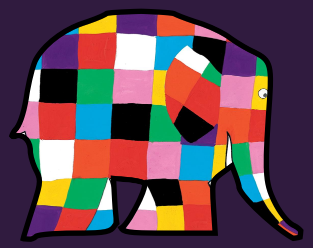
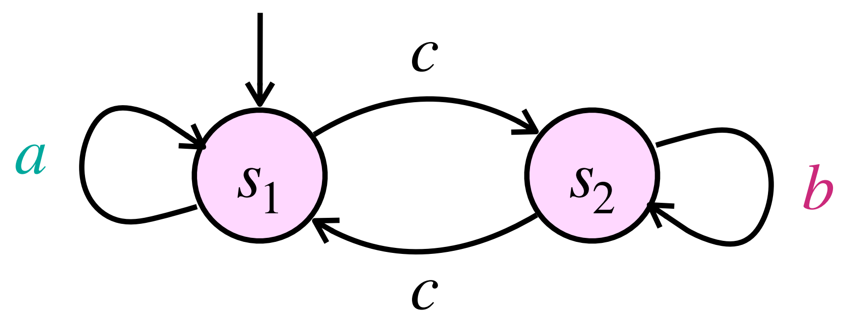
Warning



$\mathscr{M}$-determinacy requires

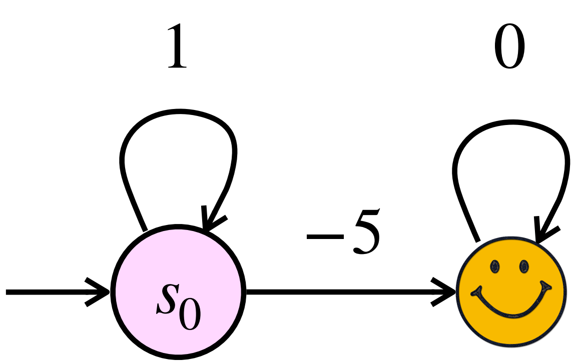
- Chromatic memory: the skeleton is based on colors
- Arena-independent memory: the same memory skeleton is used in all arenas (of the designed class)

Examples



« See infinitely often both a and b »

$\text{Büchi}(a) \wedge \text{Büchi}(b)$



« Reach the target with energy level 0 »

$\mathbf{FG}\ (\text{EL} = 0)$

Winning strategy

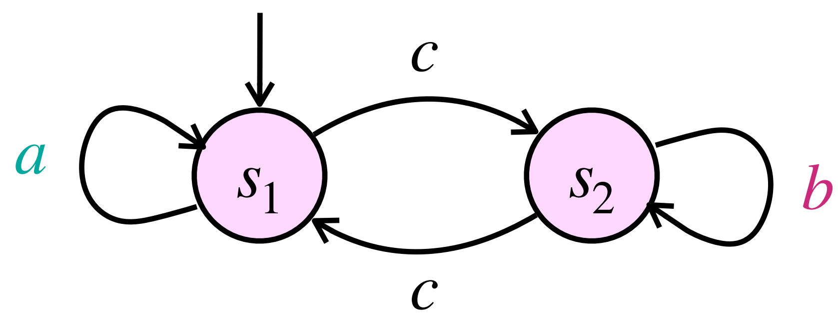
- At each visit to s_1, loop once in s_1 and then go to s_2
- At each visit to s_2, loop once in s_2 and then go to s_1
- Generates the sequence $(acbc)^\omega$

Winning strategy

- Loop five times in s_0
- Then go to the target
- Generates the sequence of colors $1\ 1\ 1\ 1\ 1\ \ -5\ 0\ 0\ 0\ldots$

These two strategies require only **finite** memory

Examples



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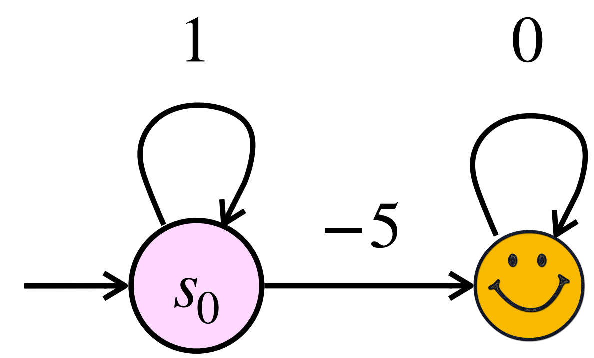

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There is an arena-independent memory based on a skeleton

Winning strategy

- Loop five times in s_0

The memory has to be arena-dependent

These two strategies require only **finite** memory

Our goal

Understand well low-memory specifications

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Memoryless / finite-memory determinacy

Is it the case that memoryless (resp. finite-memory) strategies suffice to win when winning strategies exist?

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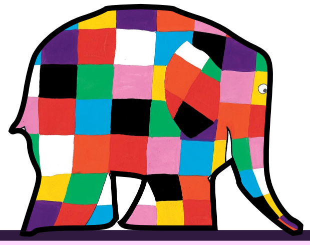
Understand well low-memory specifications



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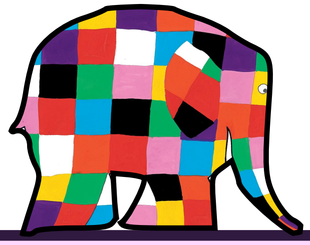

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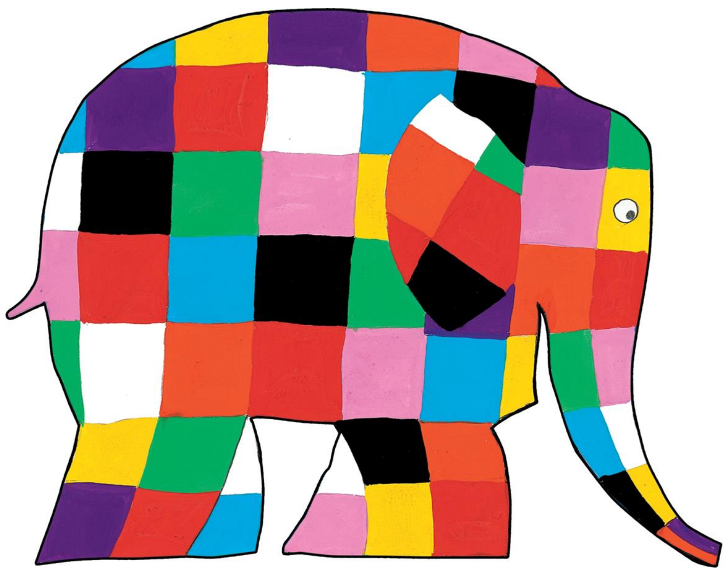
- Finite vs infinite games

Laboratoire Méthodes Formelles

Characterizing positional and chromatic finite-memory determinacy in finite games



A fundamental reference: [GZ05]

Sufficient conditions

- Sufficient conditions to guarantee memoryless optimal strategies for both players [GZ04, AR17]
- Sufficient conditions to guarantee half-positional optimal strategies [Kop06, Gim07, GK14]

[GZ05] Gimbert, Zielonka. Games Where You Can Play Optimally Without Any Memory (CONCUR'05).

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Sufficient conditions

- Sufficient conditions to guarantee memoryless optimal strategies for both players [GZ04, AR17]
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- Characterization of winning objectives ensuring **memoryless determinacy** in finite games
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Monotony and selectivity

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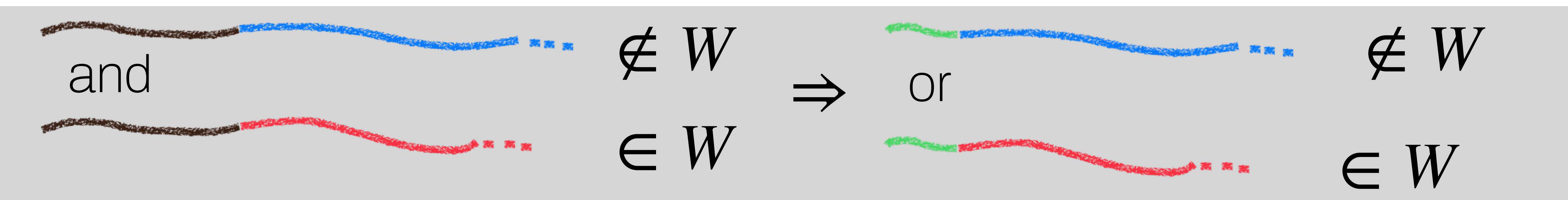

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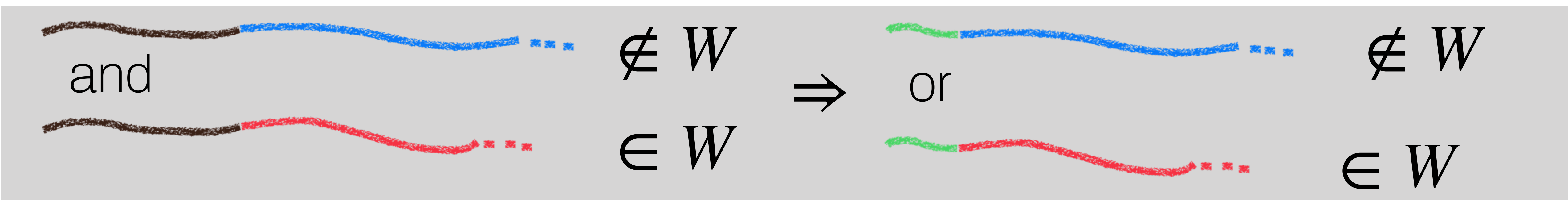
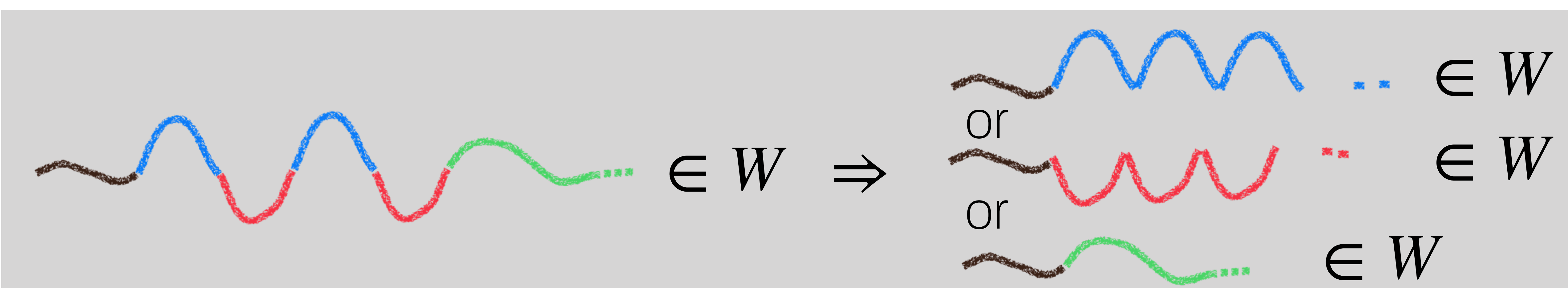


- W is **selective** whenever:



Two characterizations

Let W be an objective

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The two following assertions are equivalent:

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Why? Proof hint (1)

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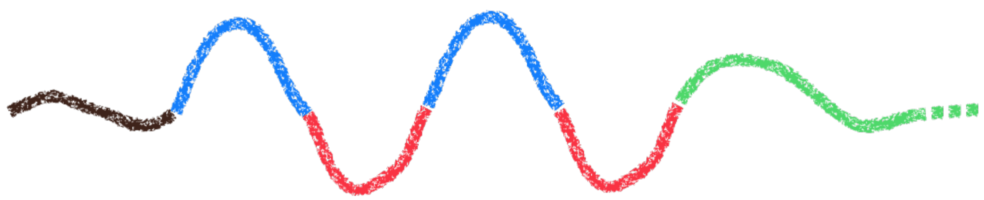
If 

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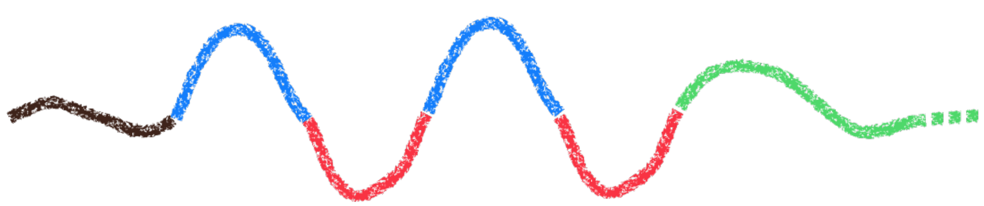

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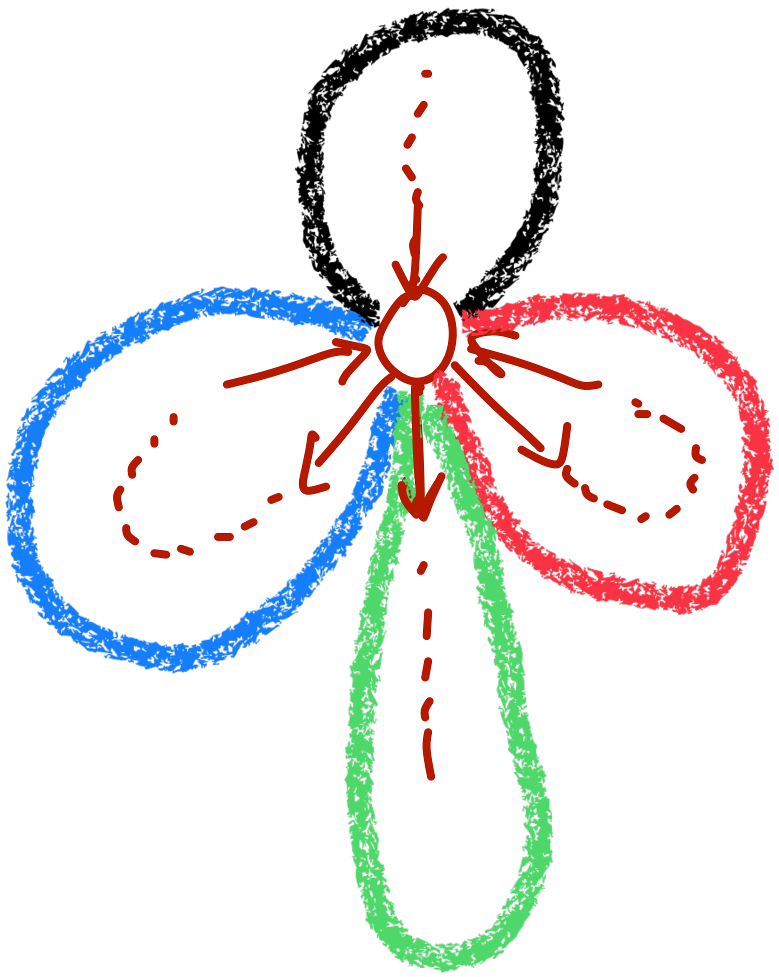

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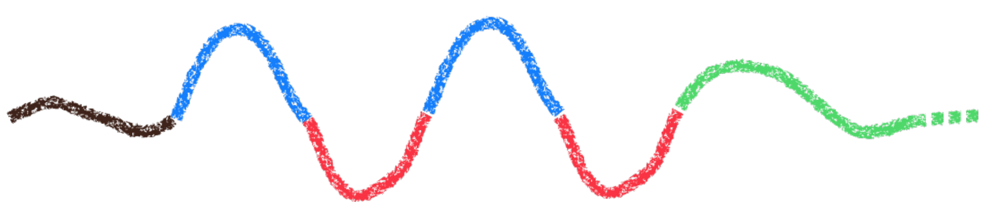
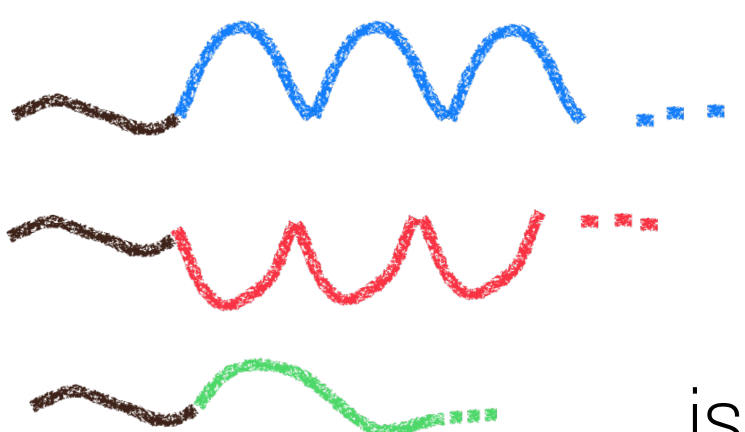
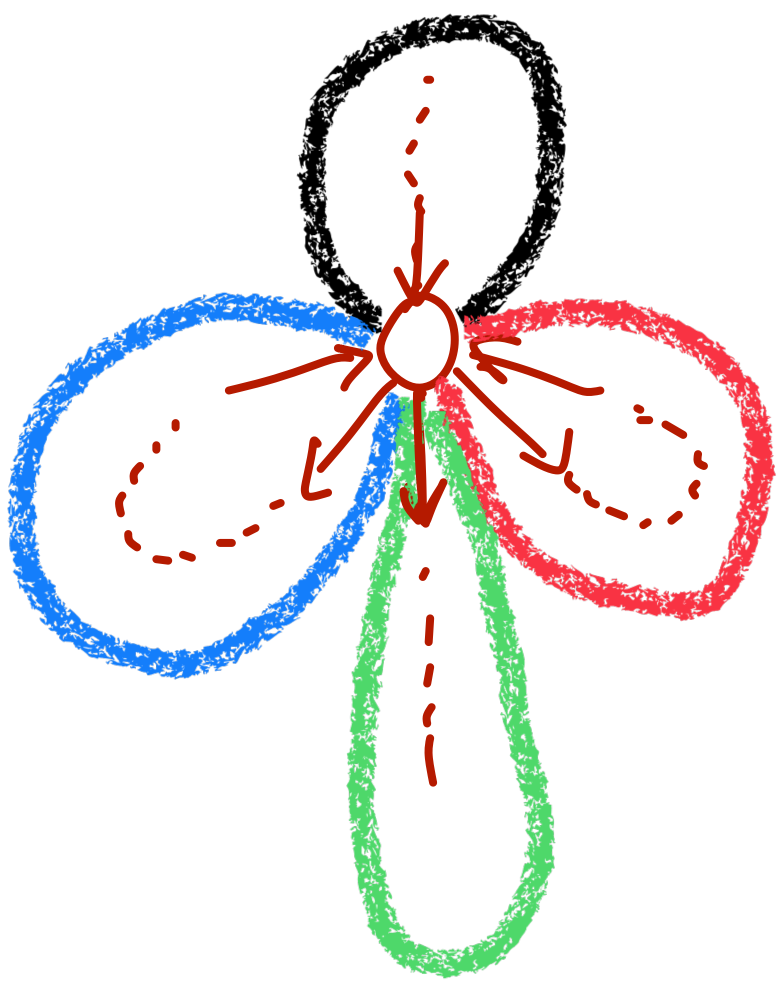
If  is winning

Then one of  is winning



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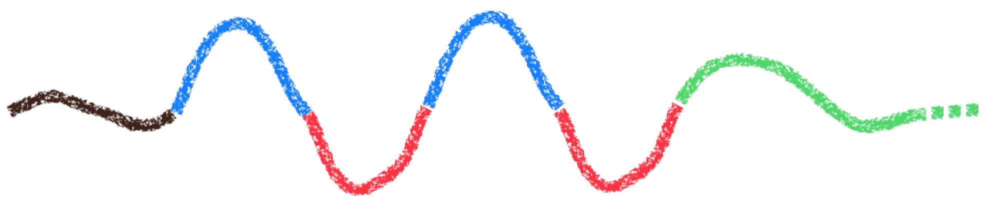
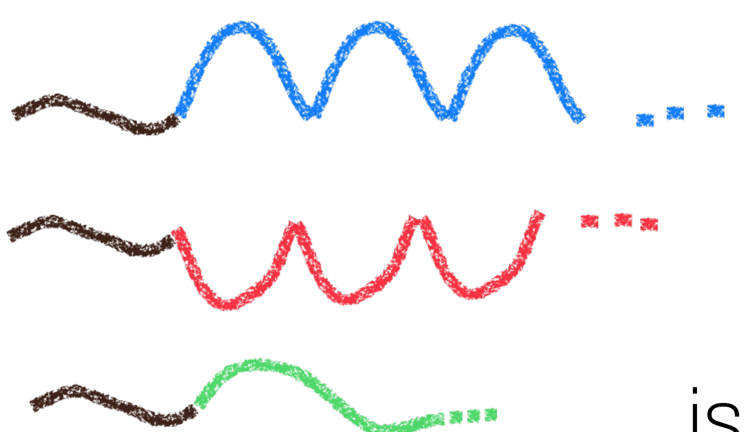
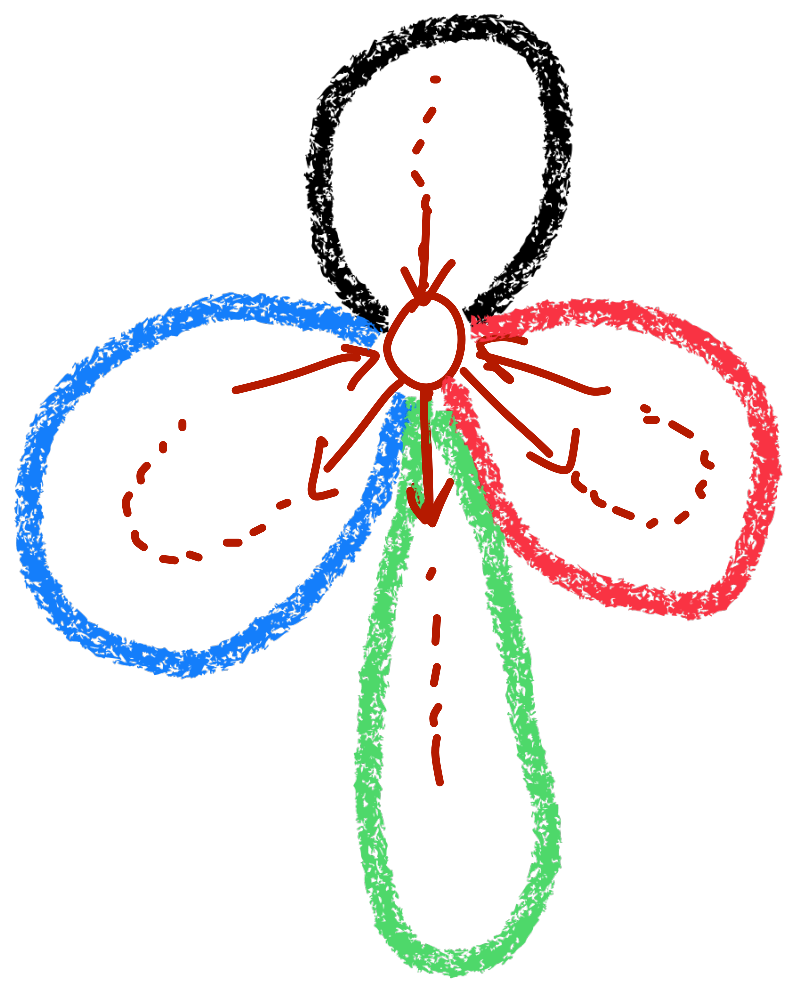
If  is winning

Then one of  is winning



W is selective

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The case of one-player arenas

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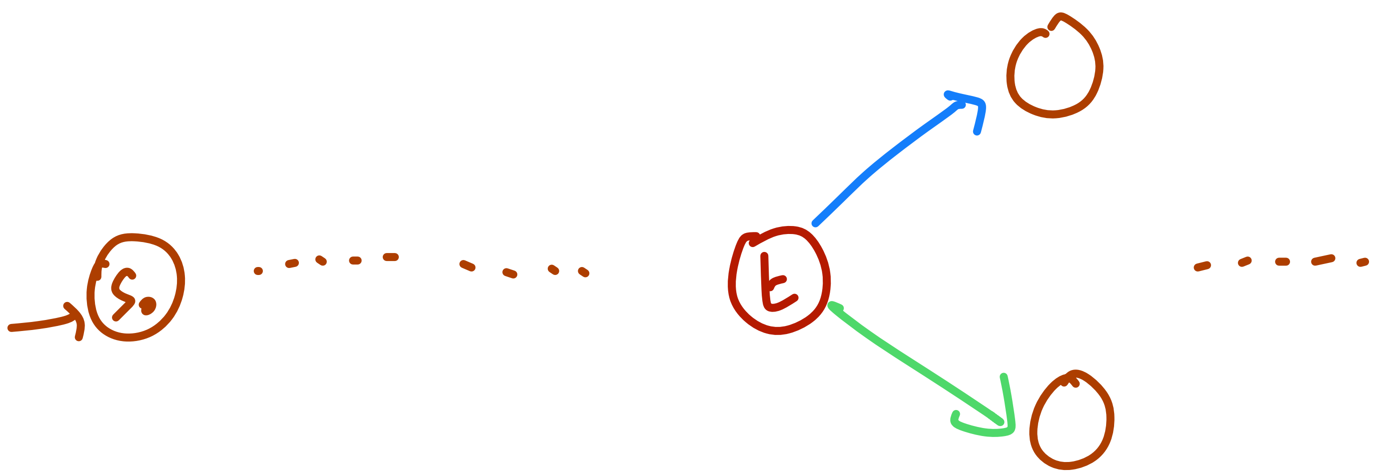

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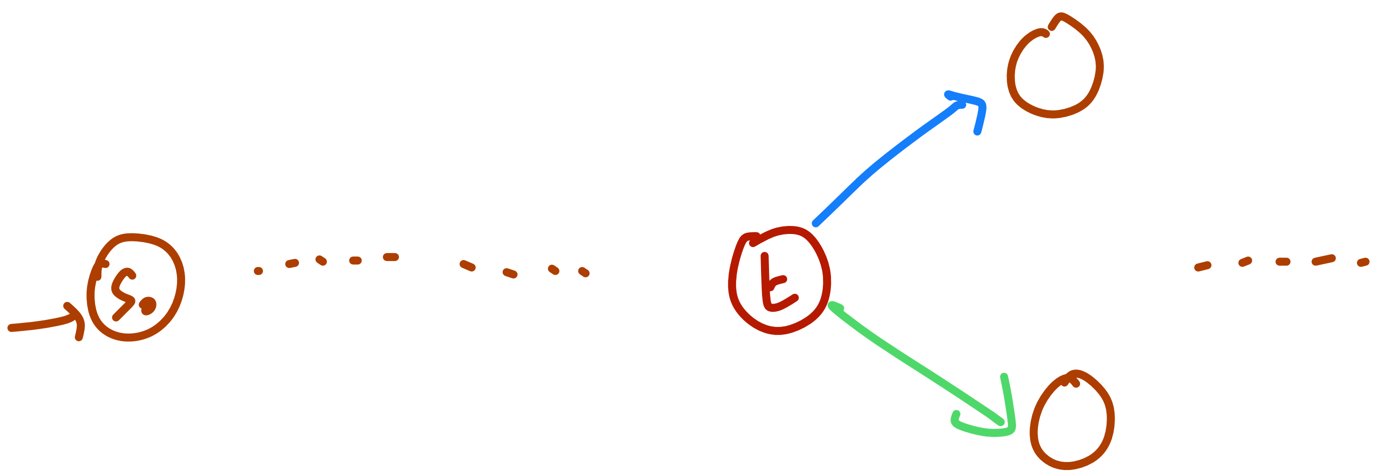

one best choice between ↗ and ↘ (monotony)

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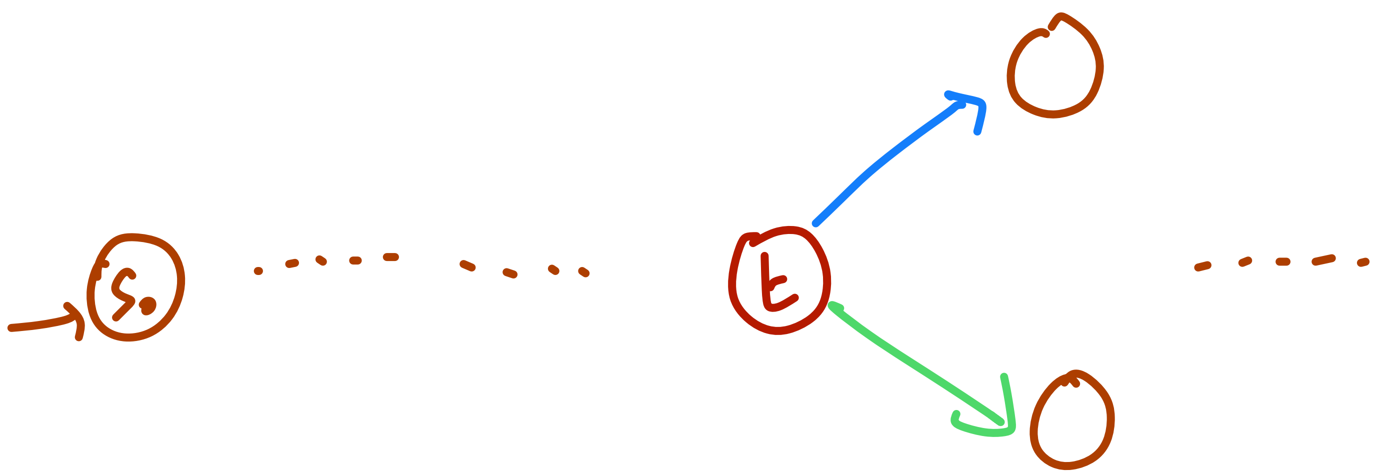
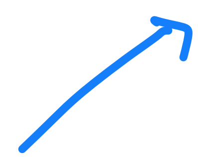
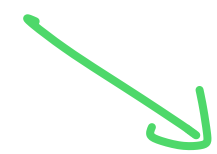
The case of one-player arenas



one best choice between  and  (monotony)
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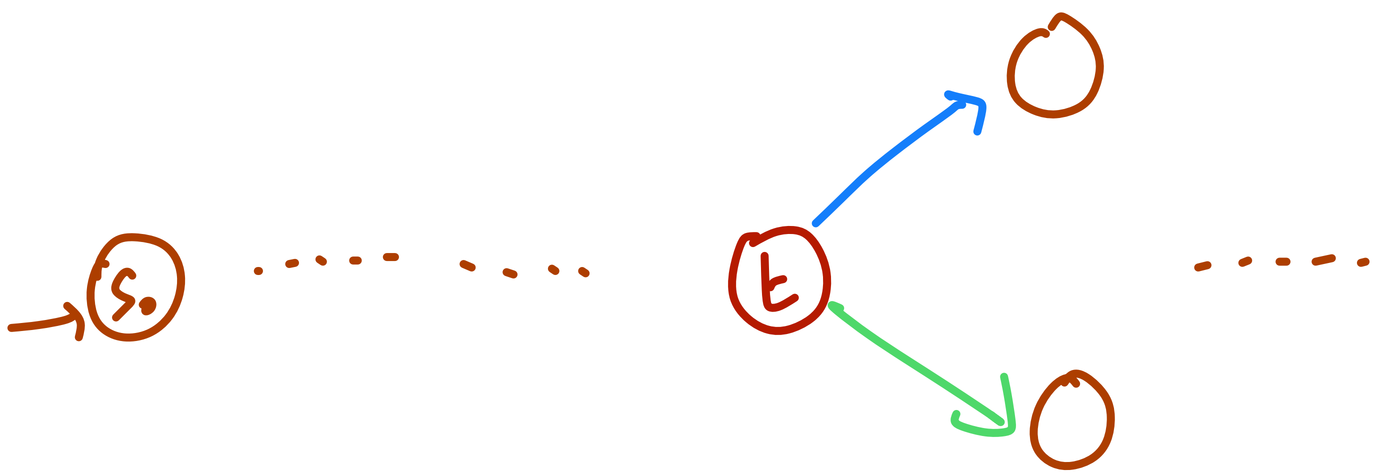
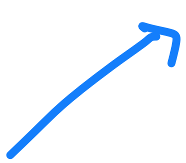
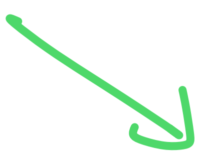


one best choice between  and  (monotony)
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No memory required at t!

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Applications

Lifting theorem

Memoryless strategies suffice for W for P_i ($i = 1,2$) in finite P_i-arenas

$$\Downarrow$$

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W is memoryless-determined in finite arenas

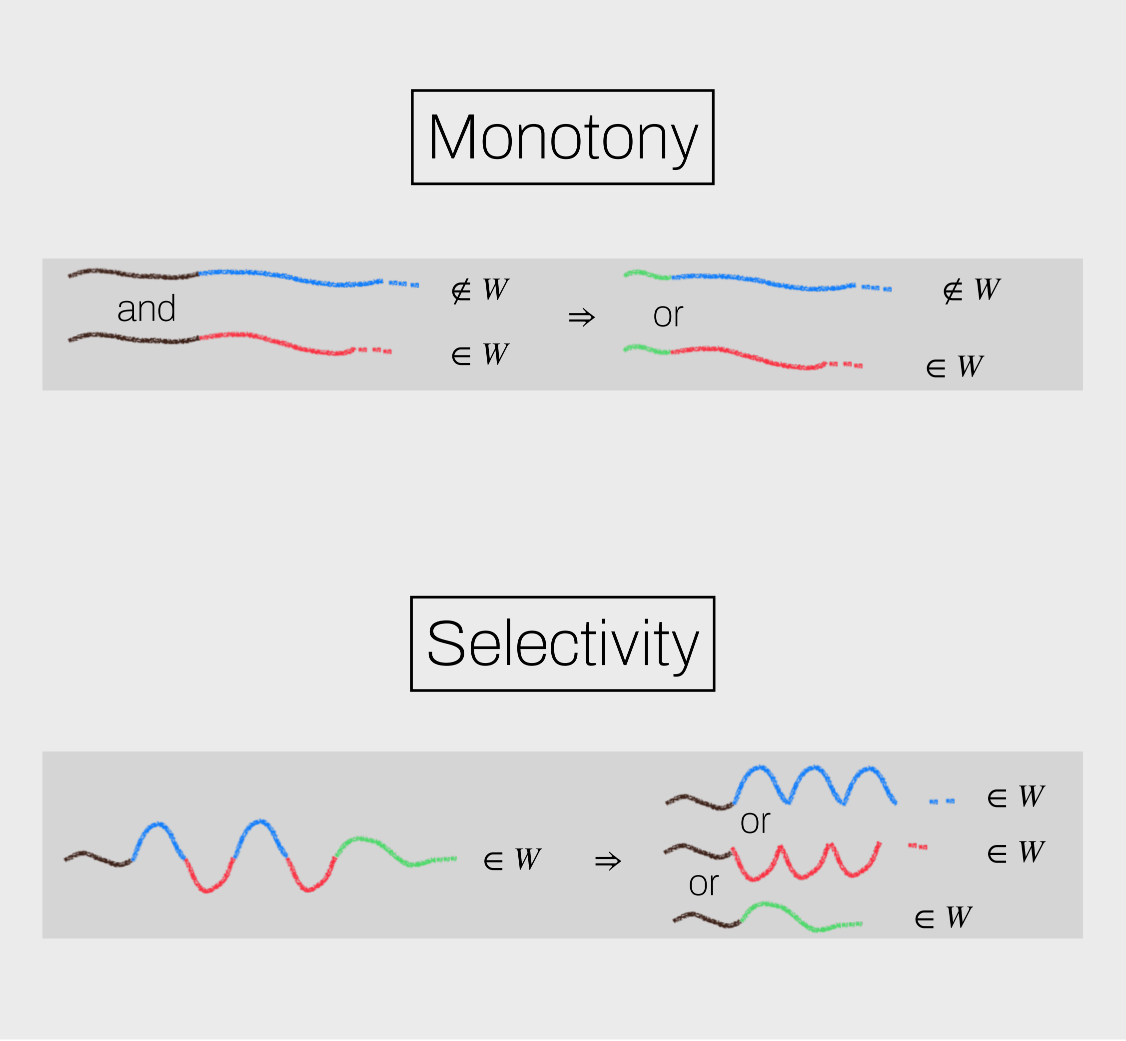
Very powerful and extremely useful in practice

- Easy to analyse the one-player case (graph reasoning)
  - Mean-payoff, average-energy [BMRLL15]
- Lift to two-player games via the theorem

[BMRLL15] Bouyer, Markey, Randour, Larsen, Laursen. Average-energy games (GandALF'15).

Discussion of examples

- Reachability, safety:
  - Monotone (though not prefix-independent)
  - Selective
- Parity, mean-payoff:
  - Prefix-independent hence monotone
  - Selective
- Average-energy games [BMRLL15]
  - Lifting theorem!!



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  - Optimal finite-memory strategies in one-player games
  - But not in two-player games!!

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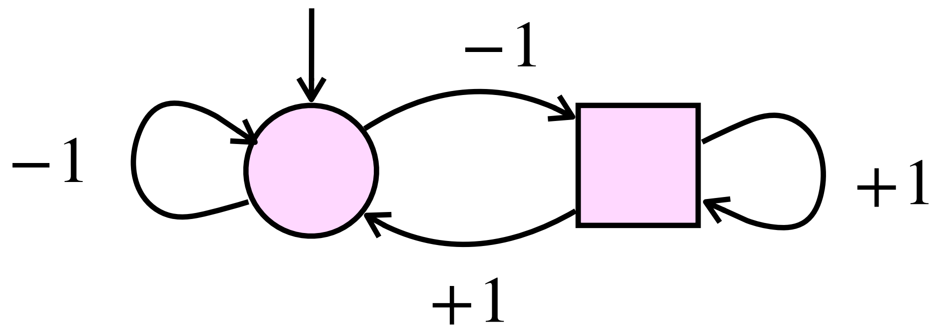
  - But not in two-player games!!



Can we lift [GZ05] to finite memory?

- No, in general
- Consider the objective W defined by

$$\liminf_n \sum_{i=1}^{n} c_i = +\infty \text{ or } \exists^\infty n \text{ s.t. } \sum_{i=1}^{n} c_i = 0$$

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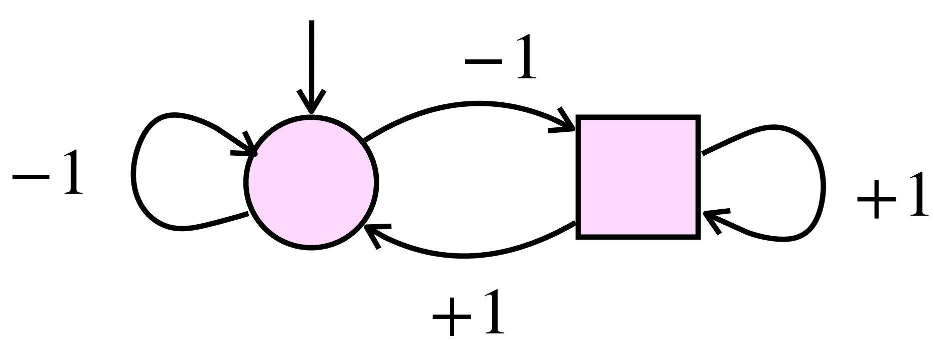
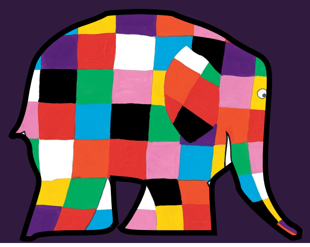


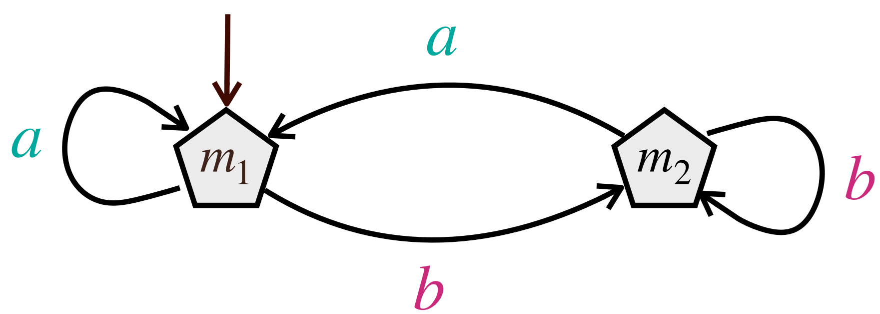
P_1 wins but requires infinite memory

Chromatic memory



Memory skeleton

$\mathcal{M} = (M, m_{\text{init}}, \alpha_{\text{upd}})$ with $m_{\text{init}} \in M$ and $\alpha_{\text{upd}} : M \times C \to M$



Not yet a strategy!

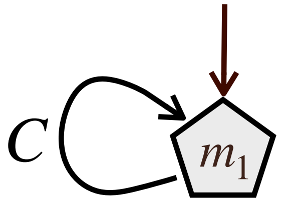
$\sigma_i : S^*S_i \to E$

Strategy with memory $\mathcal{M}$

Additional next-move function $\alpha_{\text{next}} : M \times S_i \to E$

$(\mathcal{M}, \alpha_{\text{next}})$ defines a strategy!

Remark: memoryless strategies are $\mathcal{M}_{\text{triv}}$-strategies, where $\mathcal{M}_{\text{triv}}$ is



Adding memory

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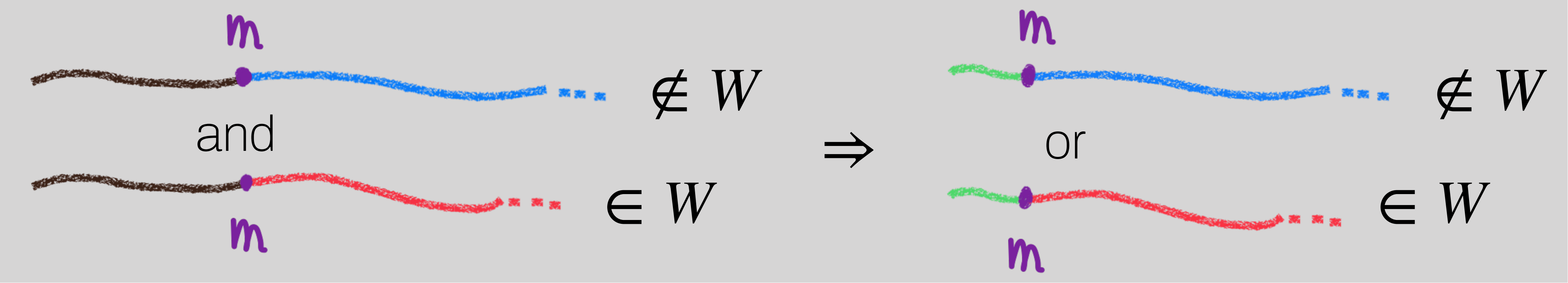
- Let W be a winning objective and $\mathscr{M}$ be a memory skeleton
- W is $\mathscr{M}$**-monotone** whenever:



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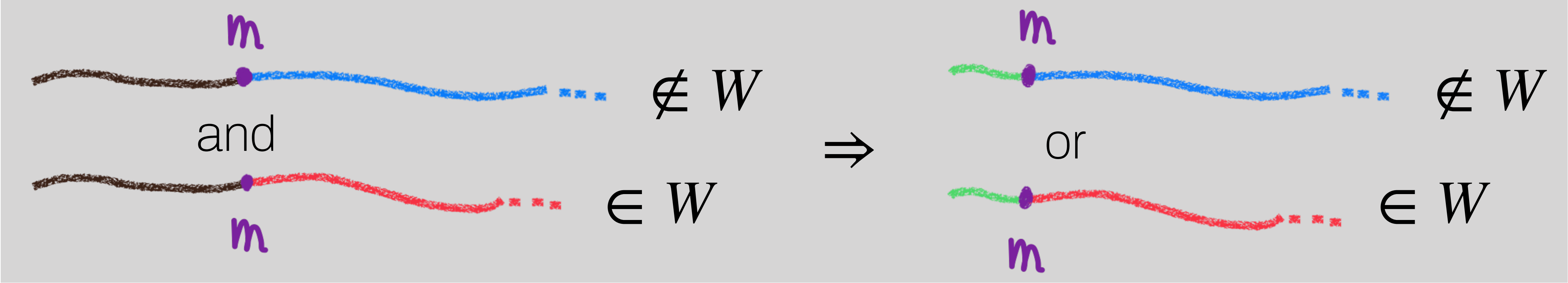
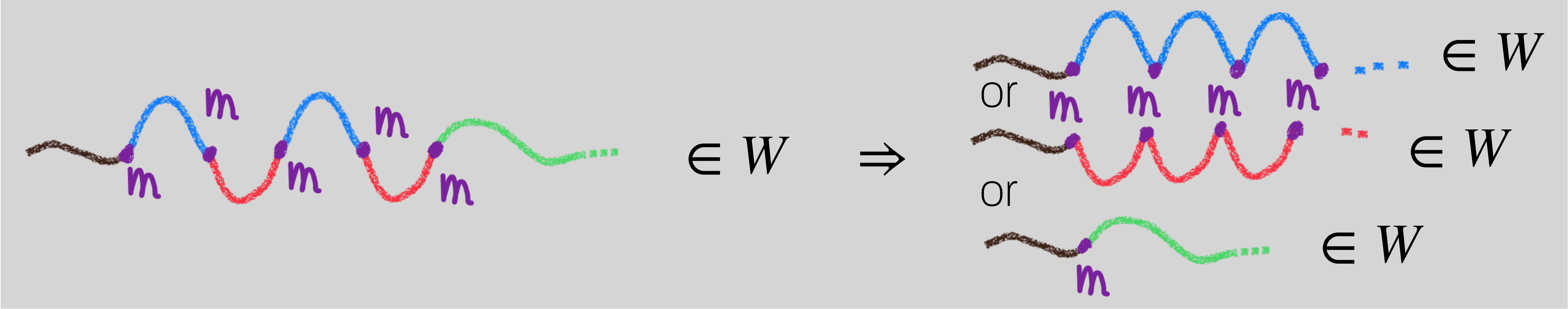


- W is $\mathscr{M}$**-selective** whenever:



Two characterizations

Let W be a winning objective and $\mathscr{M}$ be a memory skeleton

Characterization - Two-player games

The two following assertions are equivalent:

1. W is $\mathscr{M}$-determined in finite arenas;
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Characterization - One-player games

The two following assertions are equivalent:

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→ We recover [GZ05] with $\mathcal{M} = \mathcal{M}_{\text{triv}}$

Technical tool: Memory-covered arenas

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If the arena has enough information from $\mathscr{M}$,
then memoryless strategies will be sufficient

Covered arenas = same properties as product arenas

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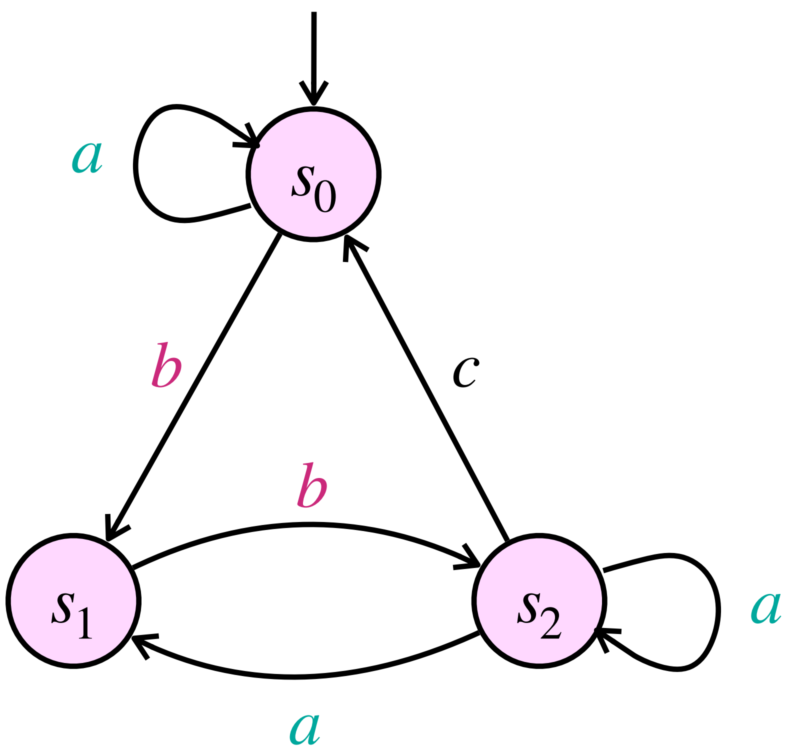

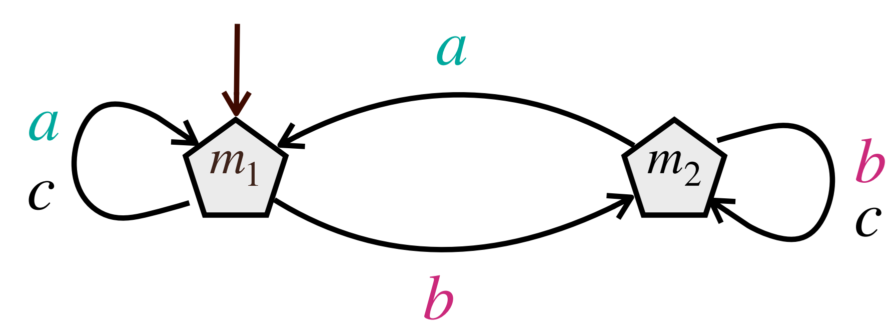

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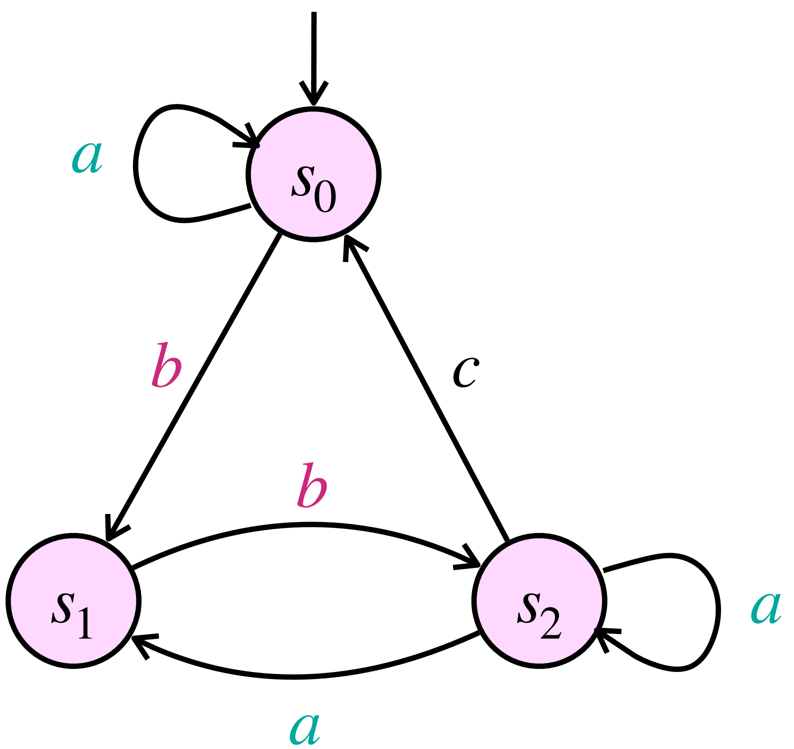
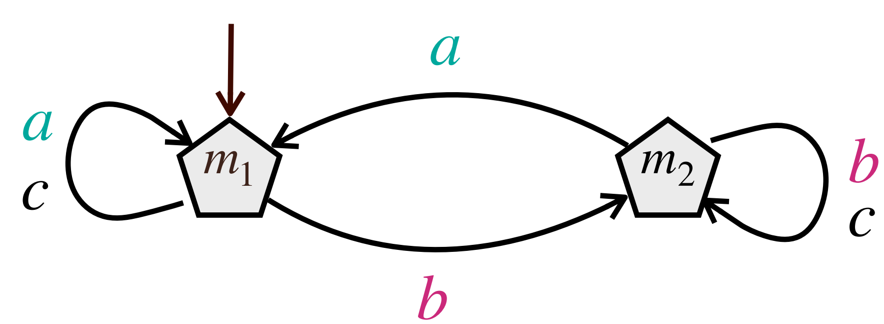
Covered arenas = same properties as product arenas





Hence one can apply a [GZ05]-like reasoning to $\mathcal{M}$-covered arenas

Applications

Lifting theorem

Strategies based on $\mathscr{M}_i$ suffice for W for P_i in finite P_i-arenas

$$\Downarrow$$

W is $(\mathscr{M}_1 \otimes \mathscr{M}_2)$-determined in finite arenas

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$$\Downarrow$$

W is $(\mathscr{M}_1 \otimes \mathscr{M}_2)$-determined in finite arenas

Very powerful and extremely useful in practice

- Easy to analyse the one-player case (graph analysis)
  - Conjunction of ω-regular objectives
- Lift to two-player games via the theorem

Example of application

W = Reach(a) ∧ Reach(b)

Example of application

W = Reach(a) $\wedge$ Reach(b)

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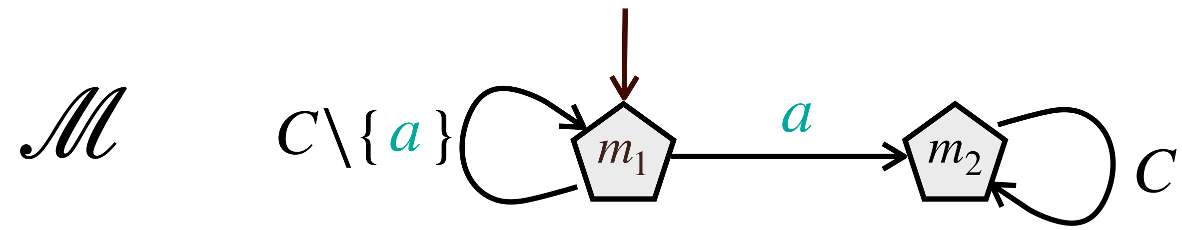

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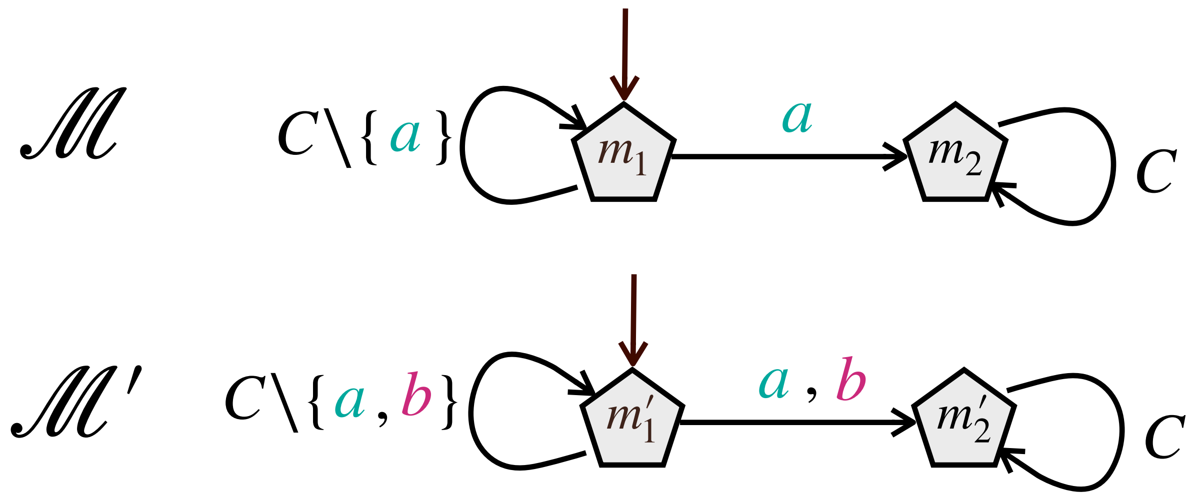

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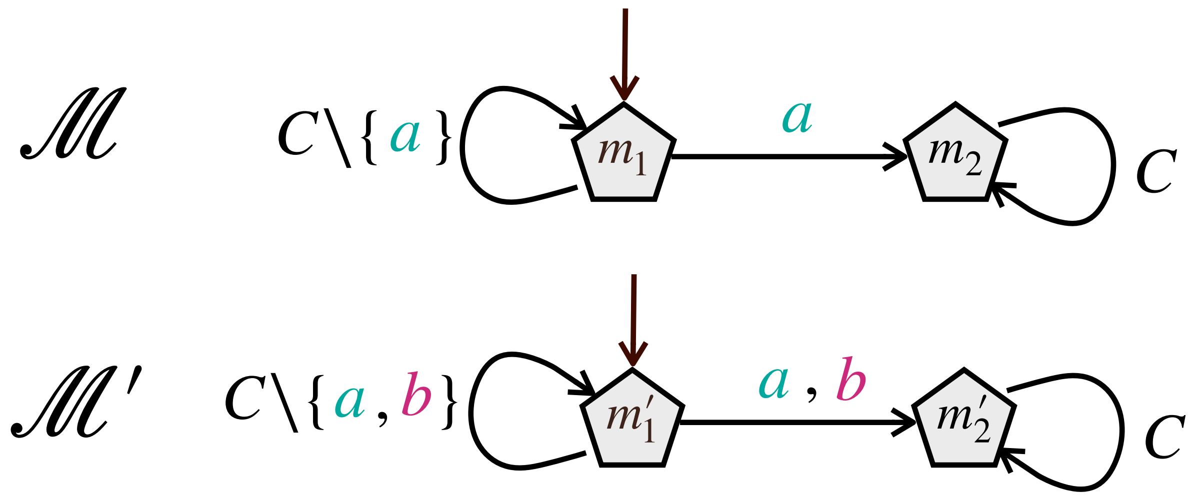

- W is $\mathcal{M}$-monotone and $\mathcal{M}'$-selective
- W^c is $\mathcal{M}$-monotone and $\mathcal{M}_{\text{triv}}$-selective

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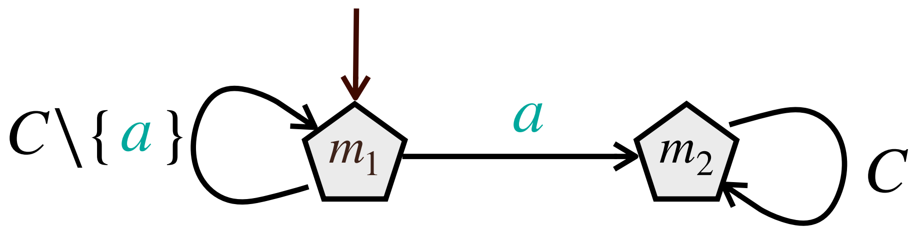
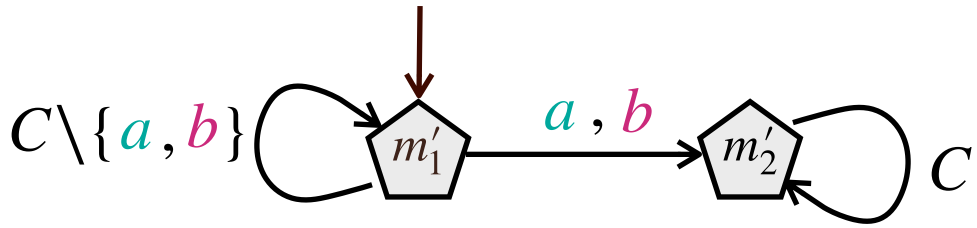
$\mathscr{M}$



$\mathscr{M}'$



- W is $\mathscr{M}$-monotone and $\mathscr{M}'$-selective
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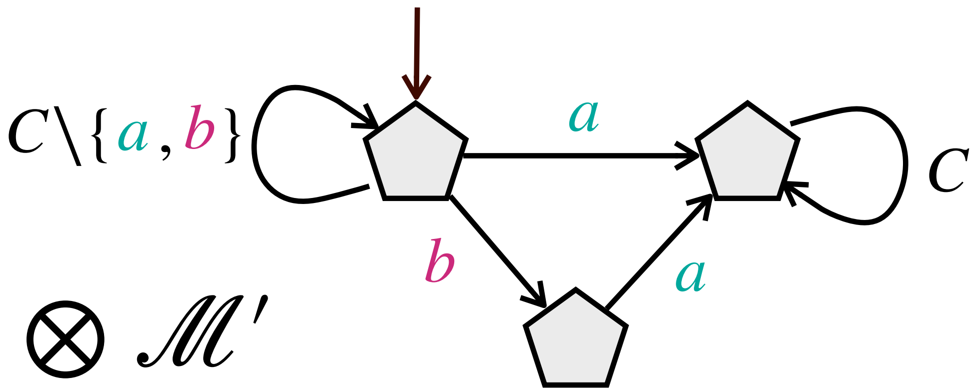

$\mathscr{M} \otimes \mathscr{M}'$

→ Memory $\mathscr{M} \otimes \mathscr{M}'$ is sufficient for both players in all finite games

Partial conclusion

Finite games

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- Complete characterization of winning objectives (and even preference relations) that ensure (chromatic) finite-memory determinacy (for both players)

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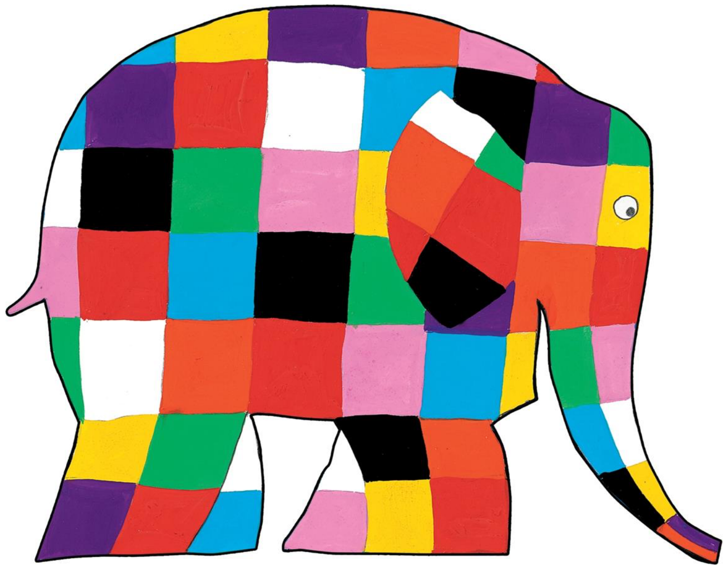
Finite games

- Complete characterization of winning objectives (and even preference relations) that ensure (chromatic) finite-memory determinacy (for both players)

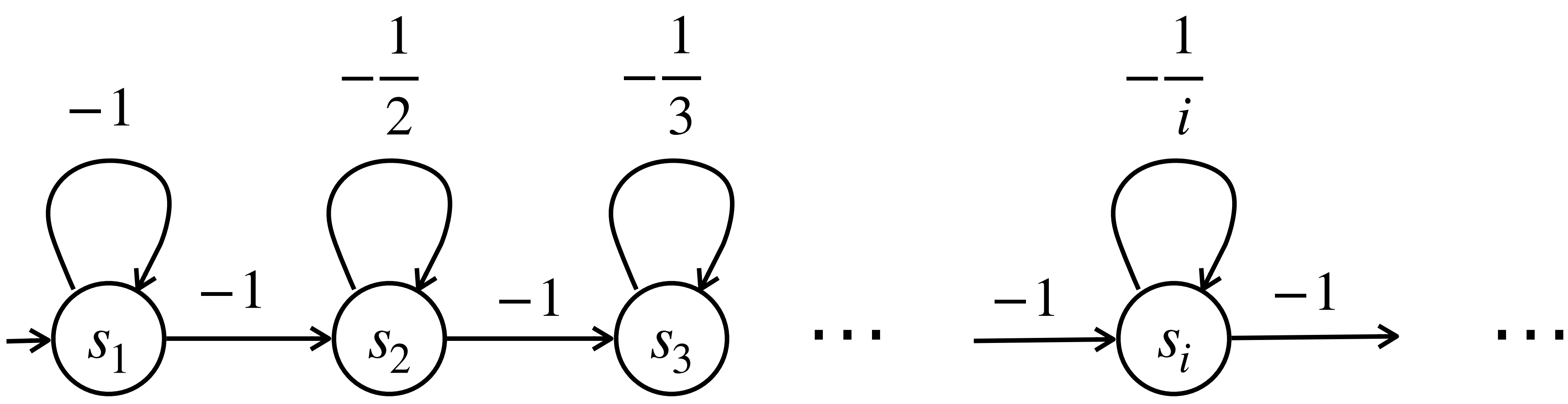
- One-to-two-player lifts
  (requires chromatic finite memory determinacy in one-player games for both players;
  ensures chromatic finite memory determinacy in two-players games for both players)

Characterizing positional and chromatic finite-memory determinacy in infinite games



The case of mean-payoff

- Objective for P_1: get non-negative (limsup) mean-payoff
- In finite games: **memoryless** strategies are sufficient to win
- In infinite games: **infinite memory** is required to win



A first insight [CN06]

- Let W be a prefix-independent objective.

[CN06] Colcombet and Niwiński. On the positional determinacy of edge-labeled games (ICALP'06).
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- Let W be a prefix-independent objective.

Characterization - Two-player games

The two following assertions are equivalent:

1. Positional optimal strategies are sufficient for W in all (infinite) games for both players;
2. W is a parity condition

   That is, there are $n \in \mathbb{N}$ and $\gamma : C \to \{0,1,\ldots,n\}$ such that
   $W = \{c_1c_2\ldots \in C^\omega \mid \limsup_i \gamma(c_i) \text{ is even}\}$

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Limitations

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Some language theory (1)

- Let $L \subseteq C^*$ be a language of finite words

Right congruence

- Given $x, y \in C^*$,

$$x \sim_L y \Leftrightarrow \forall z \in C^*, \left(x \cdot z \in L \Leftrightarrow y \cdot z \in L\right)$$

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Myhill-Nerode Theorem

- L is regular if and only if $\sim_L$ has finite index;
  - There is an automaton whose states are classes of $\sim_L$, which recognizes L.

Some language theory (2)

- Let $W \subseteq C^\omega$ be a language of infinite words

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Link with ω-regularity?

- If W is ω-regular, then $\sim_W$ has finite index;
  - The automaton $\mathcal{M}_W$ based on $\sim_W$ is a **prefix-classifier**;
- The converse does not hold (e.g. all prefix-independent languages are such that $\sim_W$ has only one element).

Characterization [BRV22]

- Let $W \subseteq C^\omega$ be an objective.

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W is finite-memory-determined if and only if W is ω-regular. Moreover, if $\mathcal{M}$ is an adapted memory skeleton for W, then W is recognized by a deterministic parity automaton built on top of $\mathcal{M} \otimes \mathcal{M}_W$.

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→ Generalizes [CN06] where both $\mathcal{M}$ and $\mathcal{M}_W$ are trivial

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$\rightarrow$ Generalizes [CN06] where both $\mathcal{M}$ and $\mathcal{M}_W$ are trivial

- The proof of $\Leftarrow$ is given by [EJ91,Zie98]

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Proof idea for $\Rightarrow$

Assume W is $\mathscr{M}$-determined. Then:

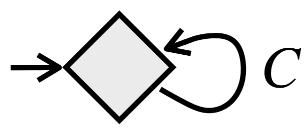
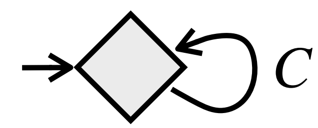
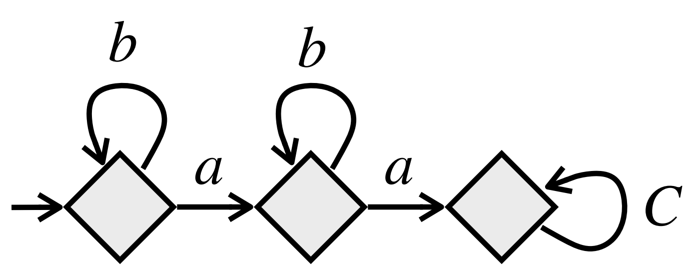
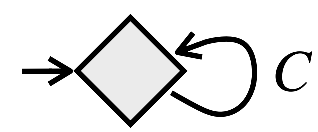
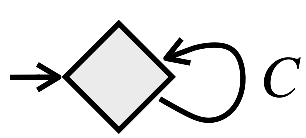
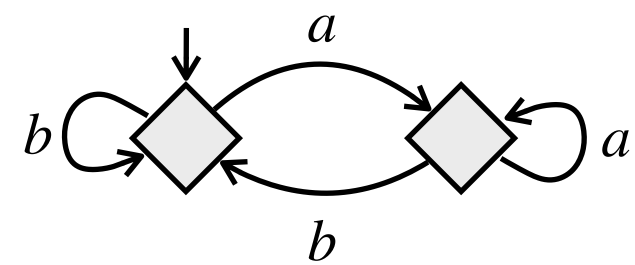
- $\mathscr{M}_W$ is finite (which implies that W is $\mathscr{M}_W$-prefix-independent);
- W is $\mathscr{M}$-cycle-consistent: after a finite word u, if $(w_i)_i$ are winning cycles of $\mathscr{M}$ (after u), then $uw_1w_2w_3\cdots$ is winning; Idem for losing cycles

$\rightarrow$ W is $(\mathscr{M} \otimes \mathscr{M}_W)$-prefix-independent and $(\mathscr{M} \otimes \mathscr{M}_W)$-cycle-consistent

$\rightarrow$ Hence W can be recognized by a DPA built on top of $\mathscr{M} \otimes \mathscr{M}_W$
(relies on ordering cycles according to how good they are for winning)

Difficult part of the proof

Examples

Objective W	**Prefix classifier $\mathscr{M}_W$**	**Memory $\mathscr{M}$**
Parity objective		
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

Examples

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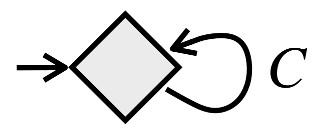
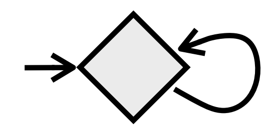
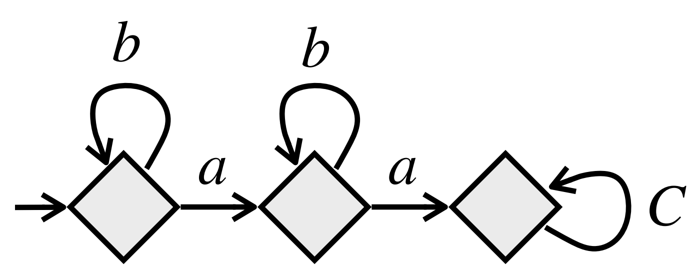
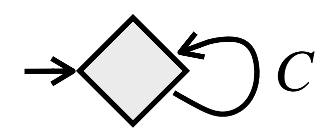
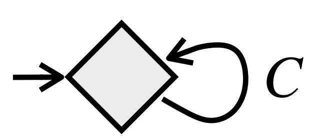
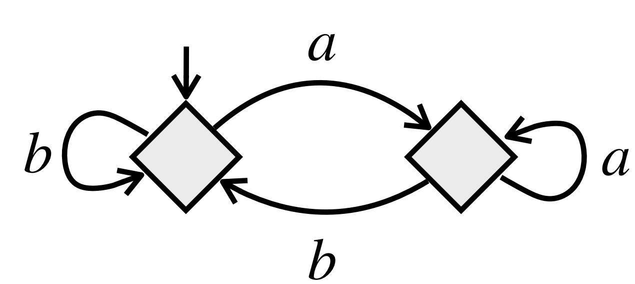
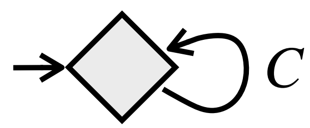
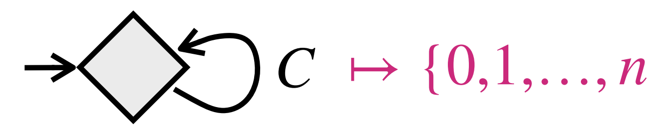
Parity objective		 $\mapsto \{0,1,\ldots,n\}$
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

Examples

Objective W | **Prefix classifier $\mathscr{M}_W$** | **Memory $\mathscr{M}$**

Parity objective





$C = \{a, b\}$
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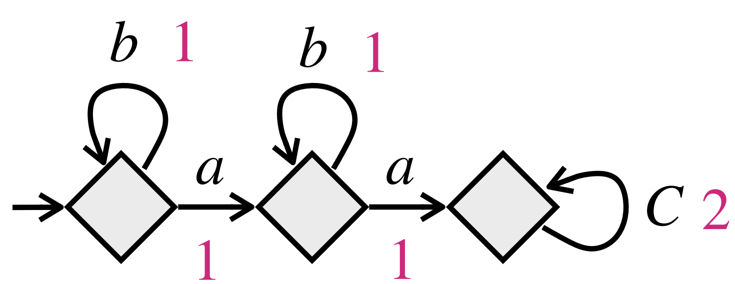

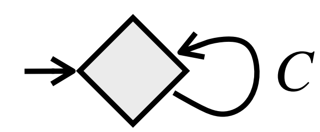

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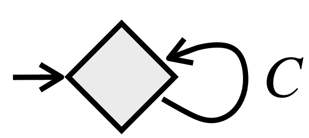
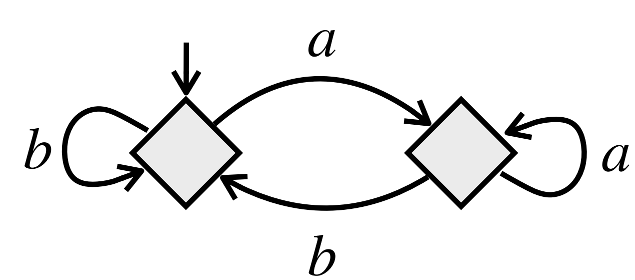
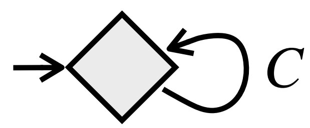
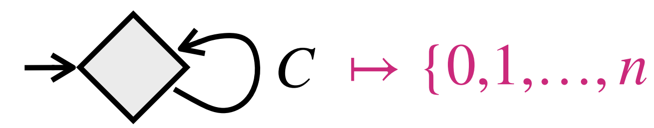
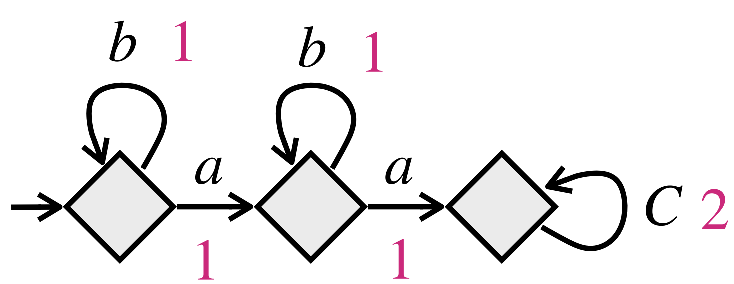
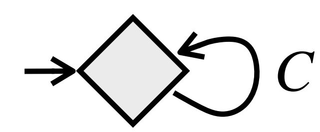
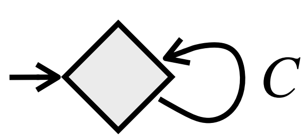
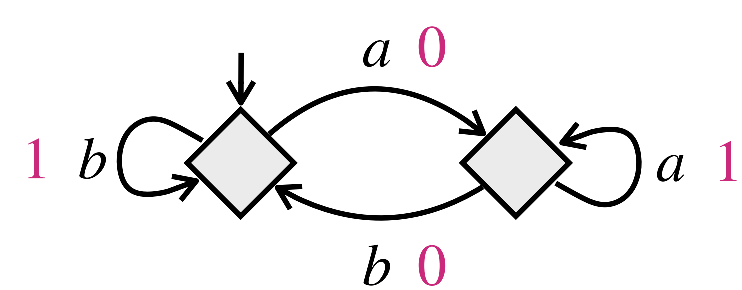




Examples

Objective W	**Prefix classifier** $\mathscr{M}_W$	**Memory** $\mathscr{M}$
Parity objective		
$C = \{a, b\}$ $W = b^*ab^*aC^\omega$		
$C = \{a, b\}$ $W = C^*(ab)^\omega$		

Corollary

Lifting theorem

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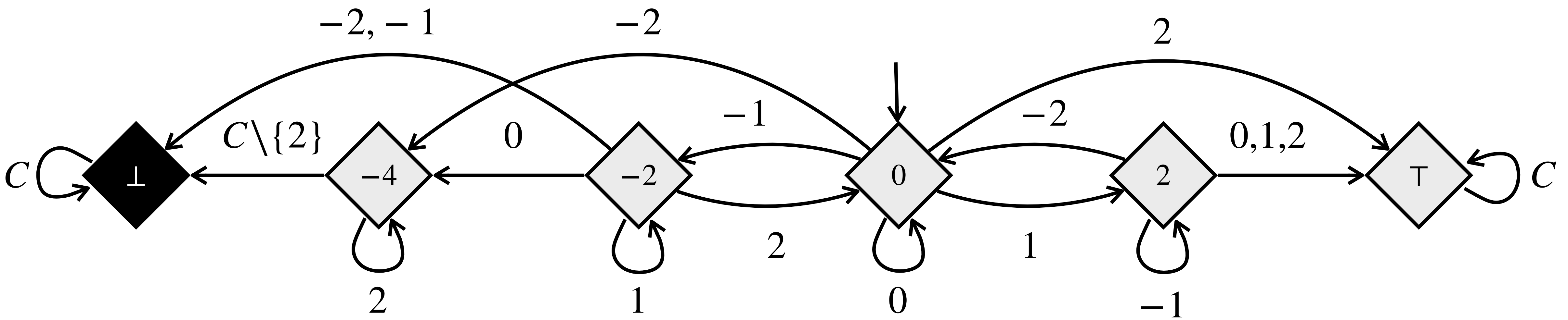

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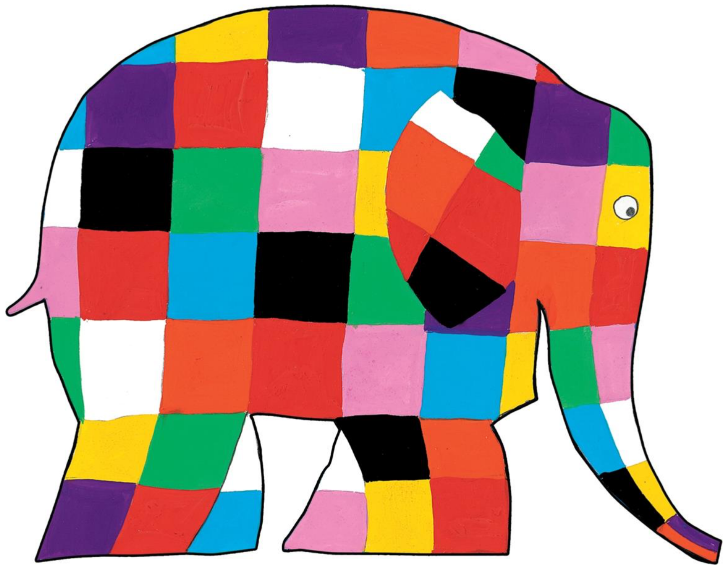
- Further questions:
  - Different results when assuming finite branching?

Going further?



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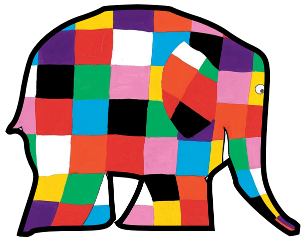

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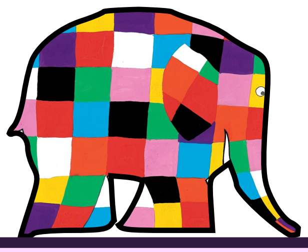

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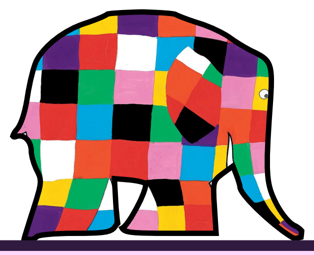

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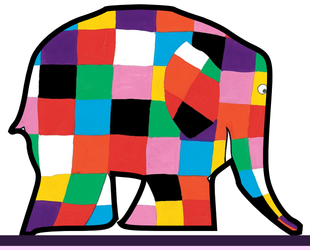

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→ Precise memory of the two players for ω-regular objectives?
(we will see it is non-trivial in general)

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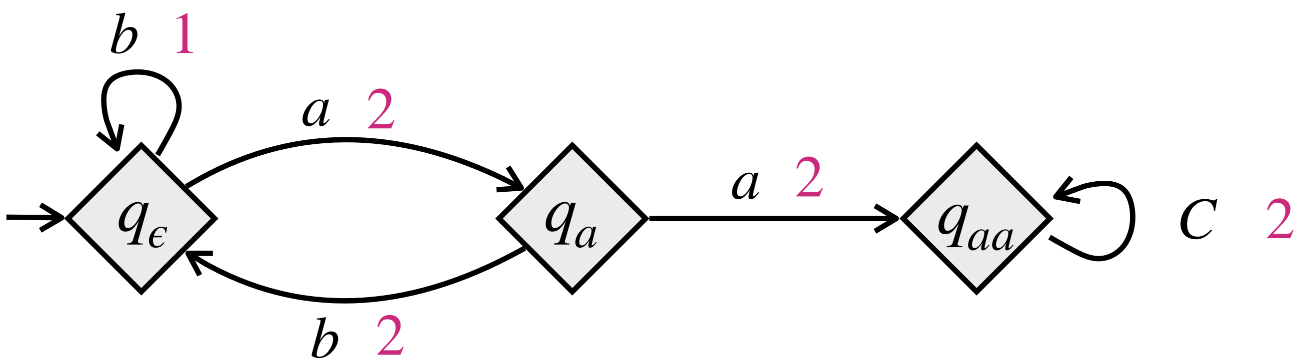

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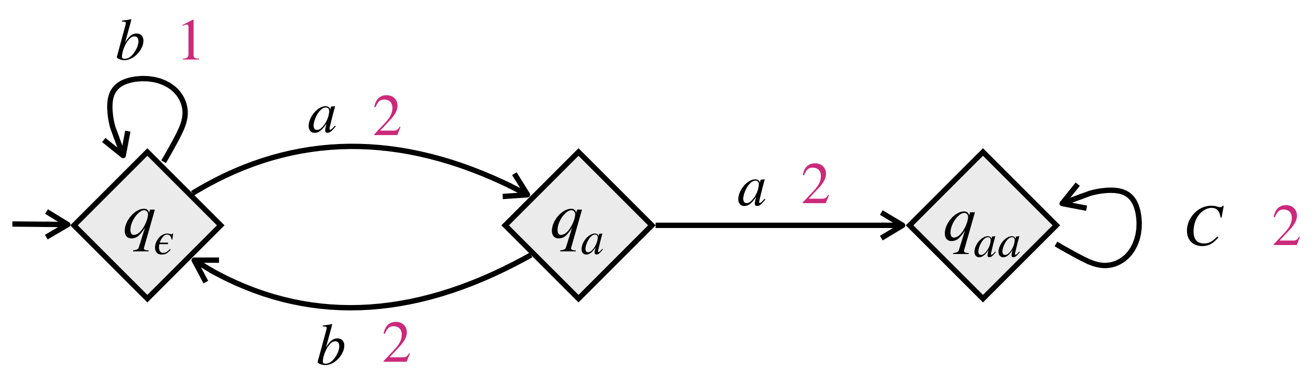

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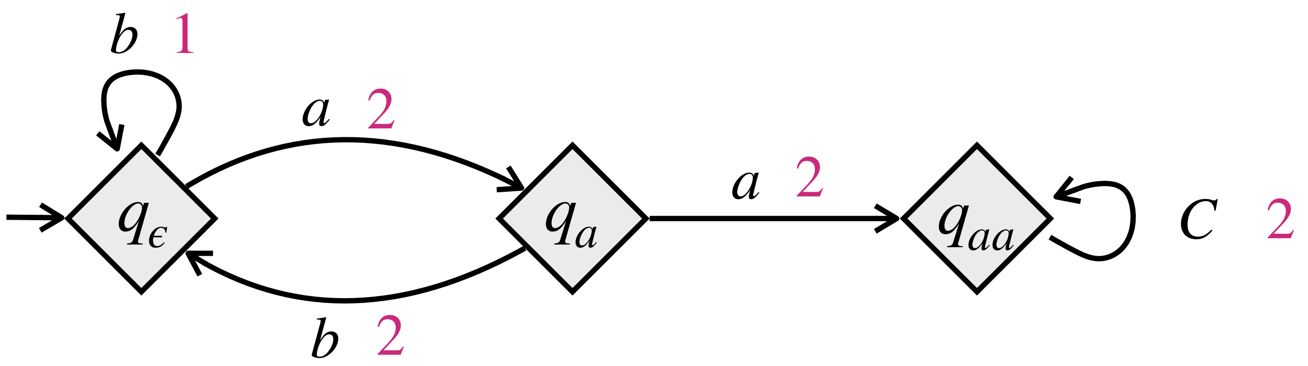

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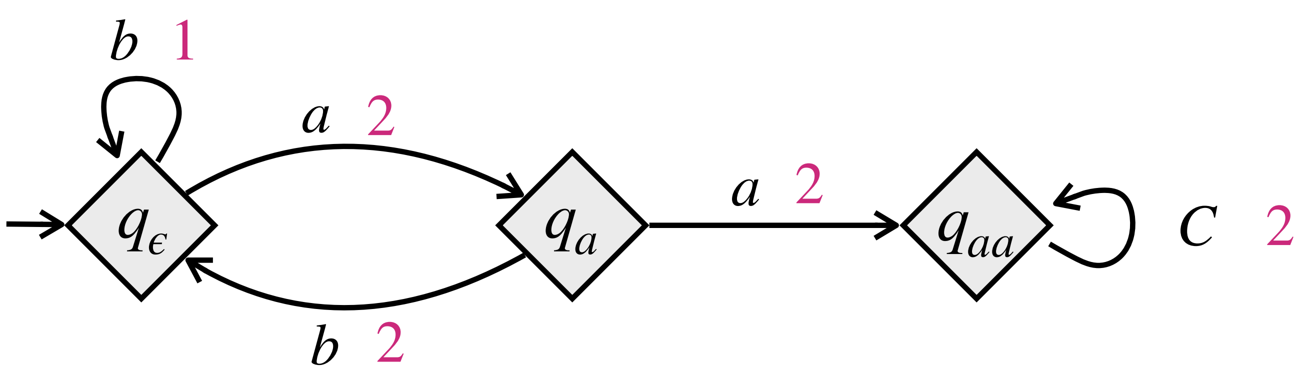

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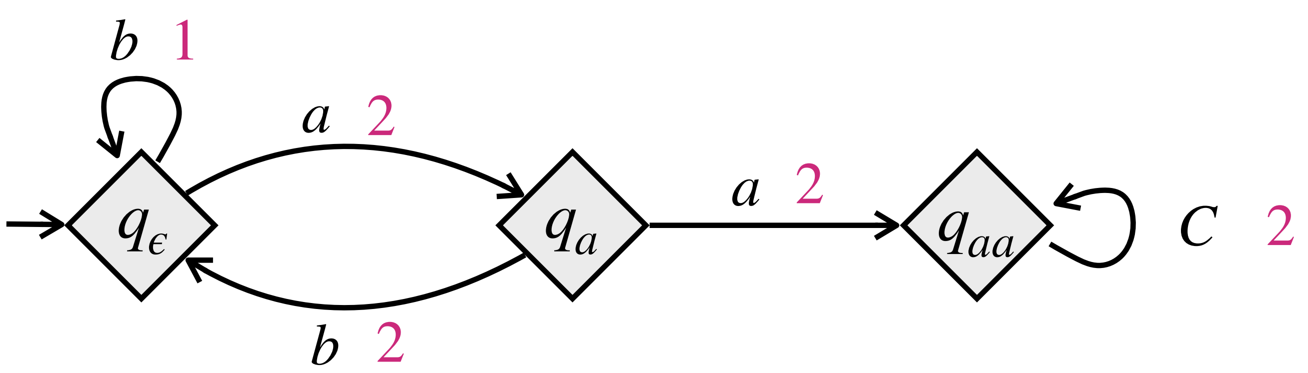

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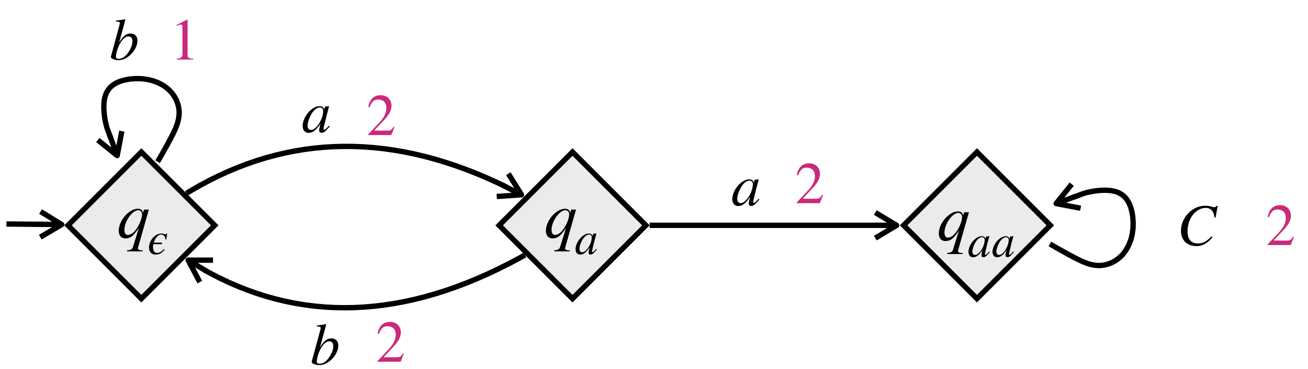

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  - P_2 requires just two states of memory: q_ϵ and q_a

The example of Muller conditions

- $\mathcal{F} \subseteq 2^C$
  $W_{\mathcal{F}} = \{w \in C^\omega \mid \{c \in C \mid \exists^\infty i \text{ s.t. } w_i = c\} \in \mathcal{F}\}$

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Chromatic memory for $W_{\mathcal{F}}$

A memory structure $\mathcal{M}$ suffices for P_1 for $W_{\mathcal{F}}$ if and only if $W_{\mathcal{F}}$ is recognized by a deterministic Rabin automaton built on top of $\mathcal{M}$ [Cas22]. It is NP-complete to decide whether there is a memory structure of size k that is sufficient to win a Muller condition.

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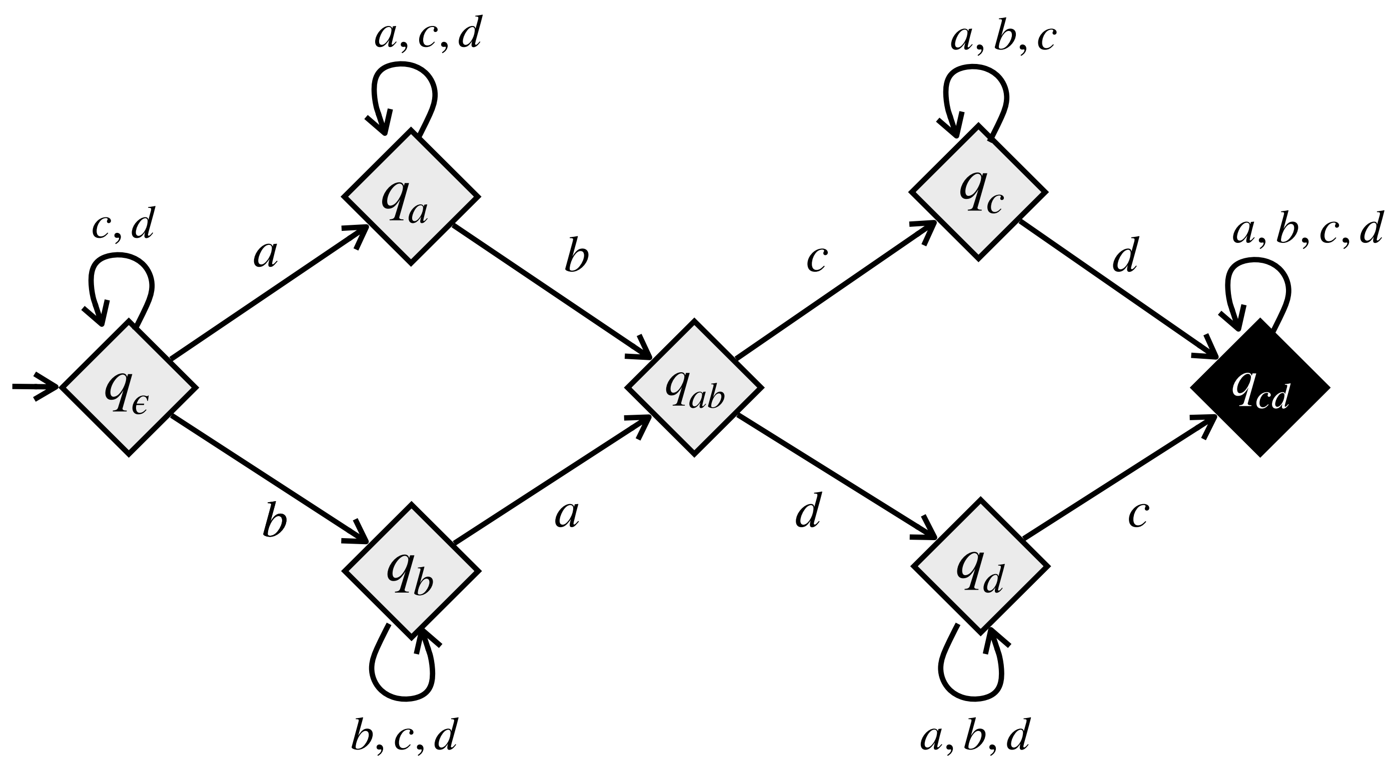
Half-positionality of W can be decided in PTIME

An objective W defined by a DBA is half-positional if and only if:

1. W is monotone;
2. W is progress consistent: if w_2 is a progress after w_1, then $w_1 w_2^\omega$ is winning;
3. W is recognized by a DBA built on top of its prefix classifier

[BCRV22] Bouyer, Casares, Randour, Vandenhove. Half-positional objectives recognized by deterministic Büchi automata (CONCUR'22)

Regular safety and reachability objectives [BFRV22]



$W =$ avoid the rightmost state

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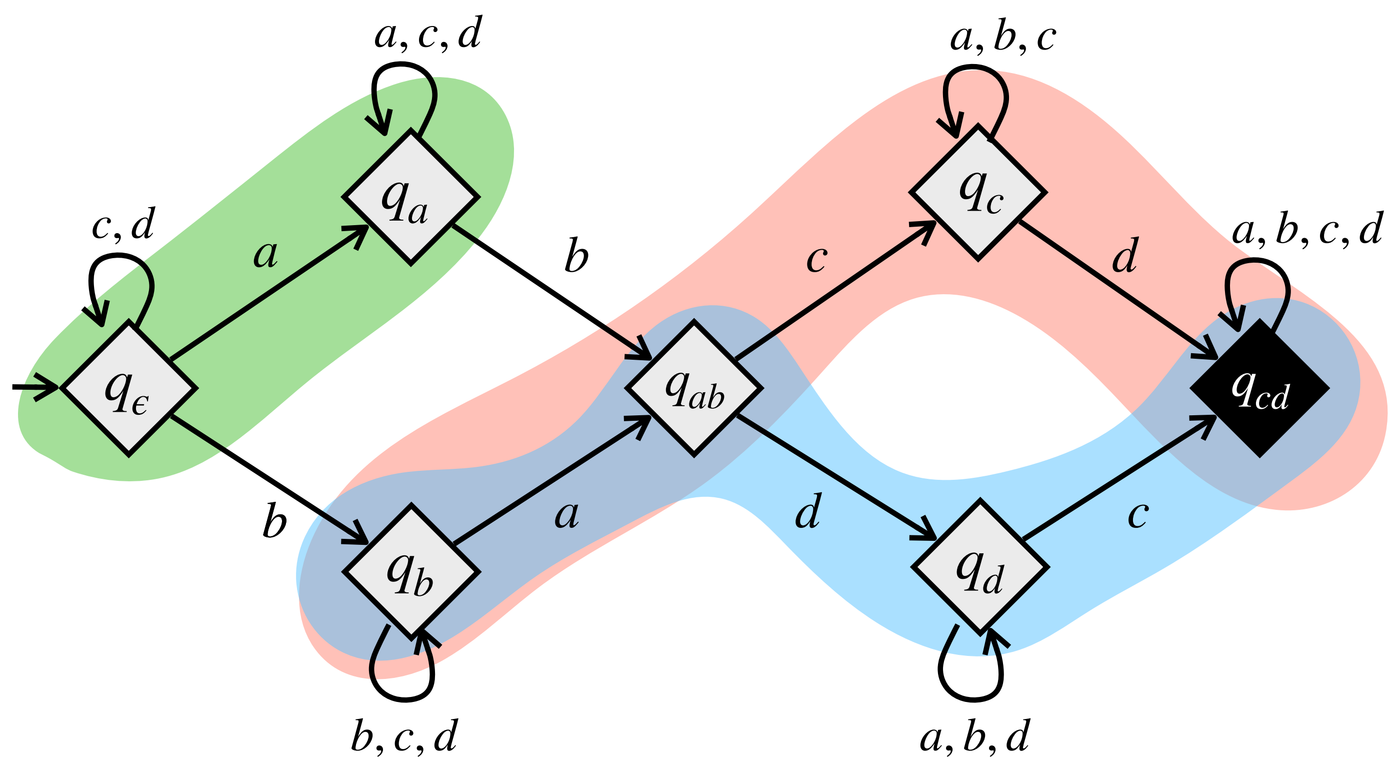

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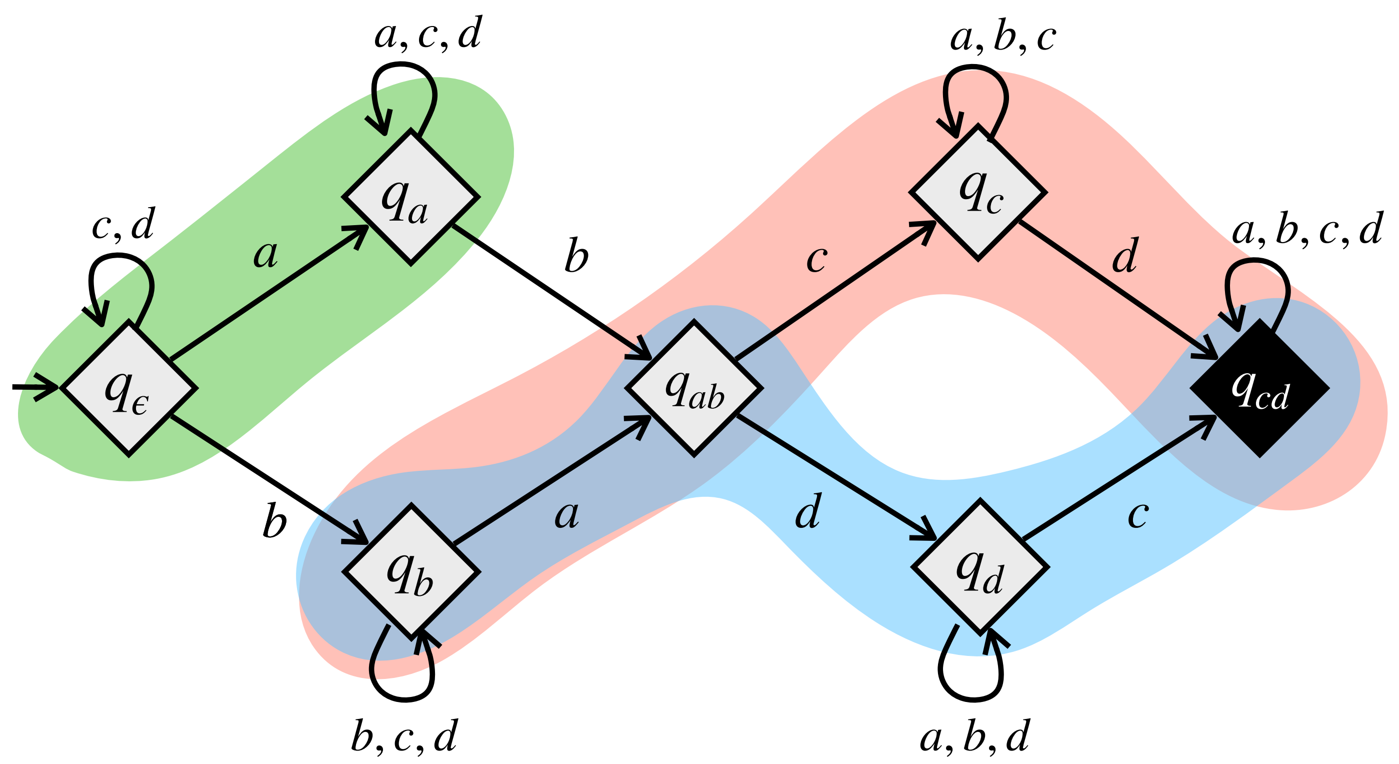

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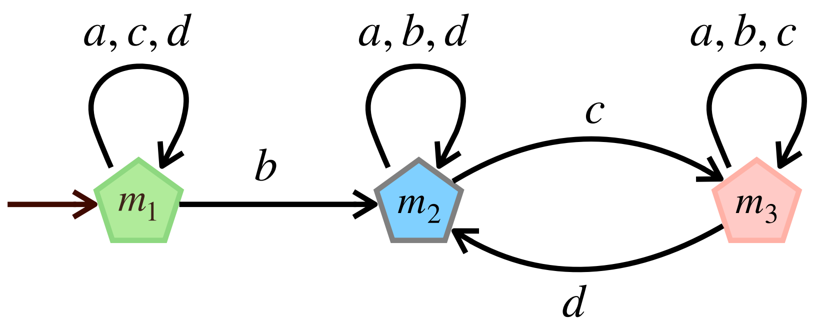

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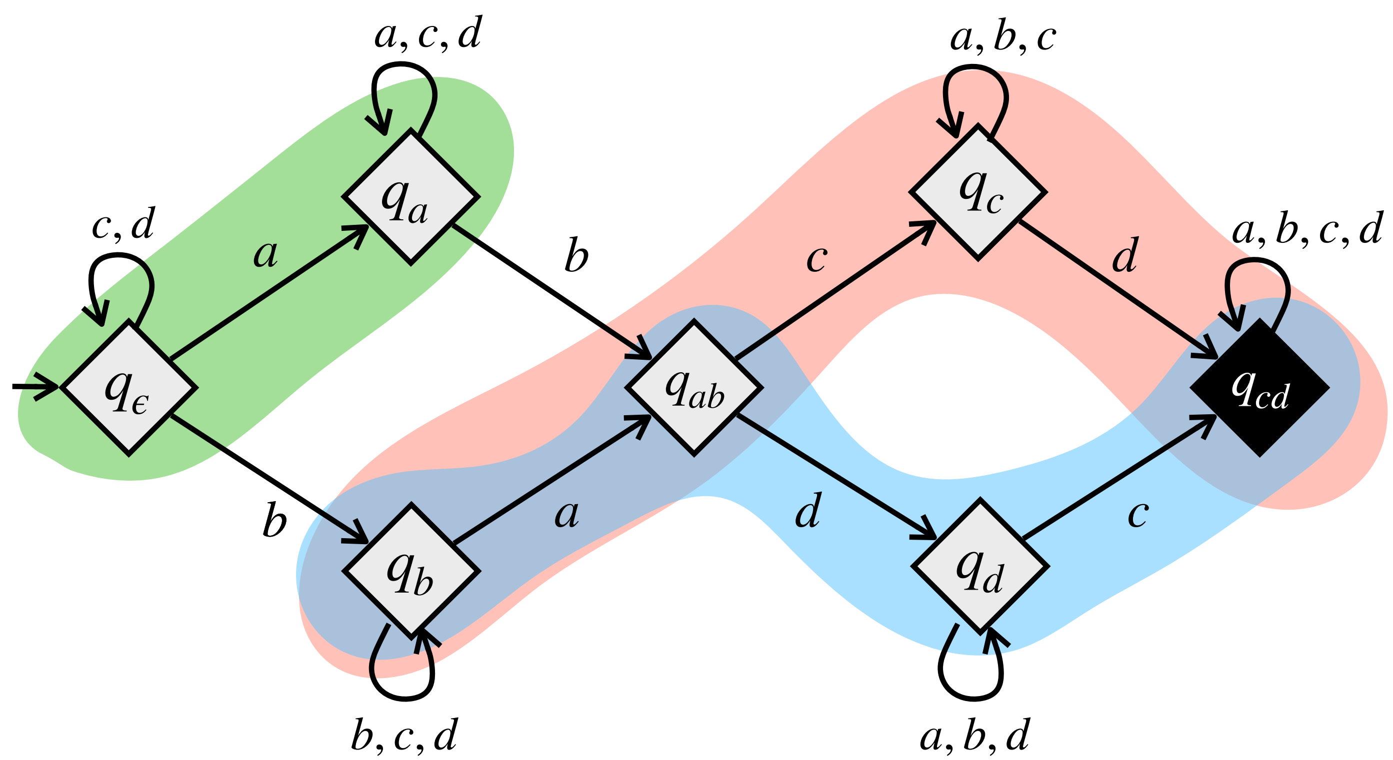

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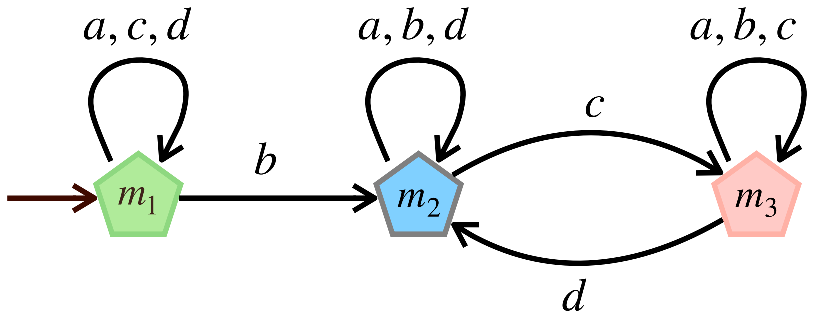

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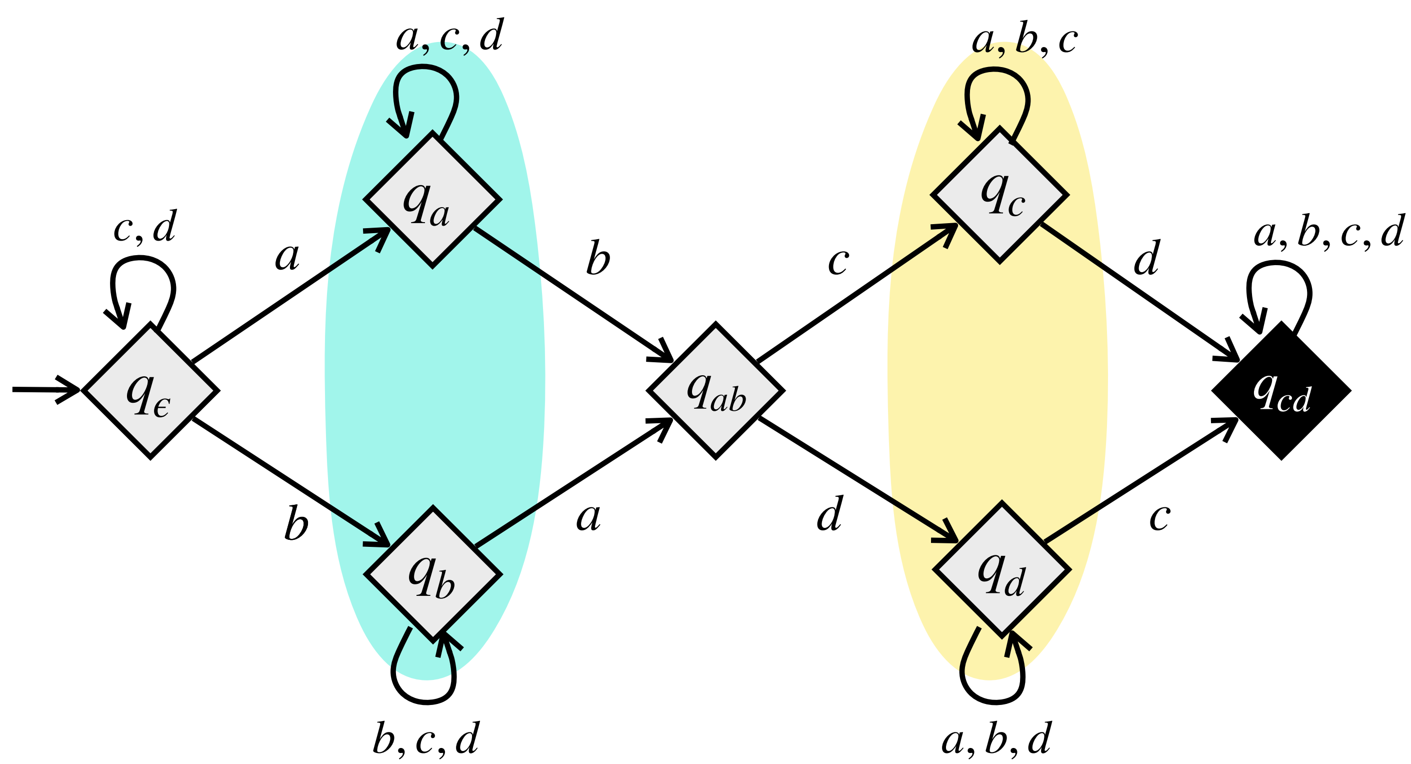
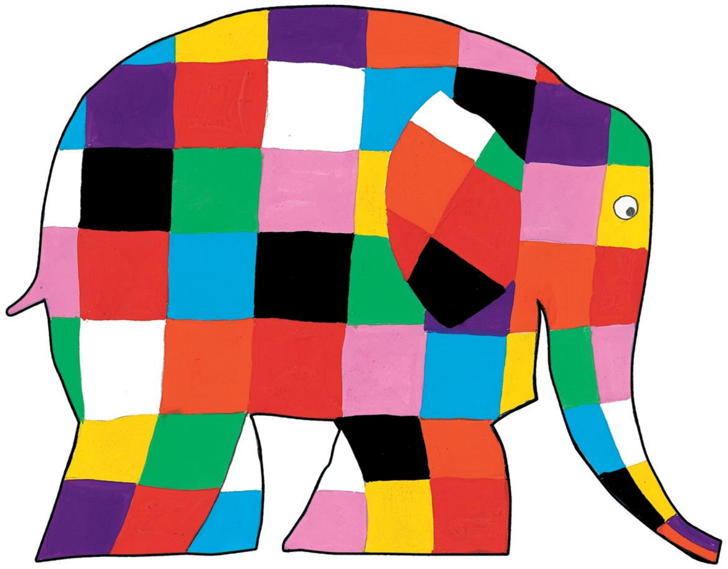
What about chaotic memory?

- Chaotic memory is more difficult to grasp
- In the previous example, only two memory states are sufficient (size of the largest antichain) [CFH14]



[CFH14] Colcombet, Fijalkow, Horn. Playing safe (FSTTCS'14)

Conclusion



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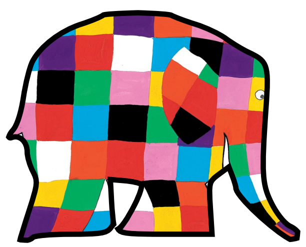

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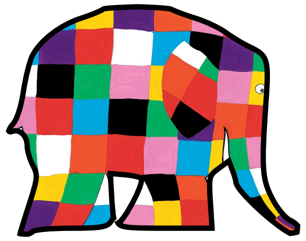

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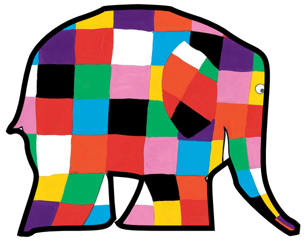

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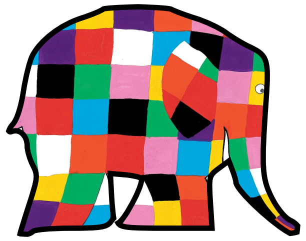

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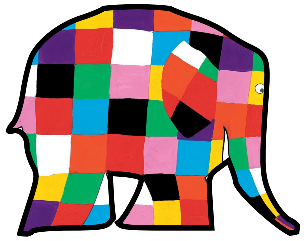

Quite active area of research

[CCL22] Casares, Colcombet, Lehtinen.On the size of good-for-game Rabin automata and its link with the memory in Muller games (ICALP'22)

[Ohl22] Ohlmann. Characterizing positionality in games of infinite duration over infinite graphs (LICS'22)